

Solutions, Section 9.4-5 (Modeling)

Throughout, dotted variables will mean the derivative with respect to time (common in physics).

1. Exercise 1, Section 9.4:

$$\begin{aligned}\dot{x} &= x(1.5 - x - 0.5y) \\ \dot{y} &= y(2 - y - 0.75x)\end{aligned}$$

Find the critical points, and at each compute the Jacobian matrix for classification. For extra practice, we should be able to compute the eigenvalues and eigenvectors as well. I will provide the plots here as well (you do not have to sketch them).

The equilibria are and the Jacobian matrix are:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 7/5 \end{bmatrix} \quad J = \begin{bmatrix} 1.5 - 2x - 0.5y & -0.5x \\ -0.75y & 2 - 2y - 0.75x \end{bmatrix}$$

We isolate and analyze the solution close to each of the equilibria:

- At the origin, the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_{1,2} = 1.5, 2 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is a classic SOURCE, with eigenvectors along the x - and y -axes.

- At $(0, 2)$, the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} 0.5 & 0 \\ -1.5 & -2 \end{bmatrix} \quad \lambda_{1,2} = 0.5, -2 \quad \mathbf{v}_{1,2} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is a SADDLE, with the stable direction along the y -axis (which is expected).

- At $(3/2, 0)$, the local matrix is again given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} -3/2 & -3/4 \\ 0 & 7/8 \end{bmatrix} \quad \lambda_{1,2} = -3/2, 7/8 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -19 \end{bmatrix}$$

This is another SADDLE, with the stable direction now along the x -axis (which is expected).

- At $(4/5, 7/5)$, the local matrix is again given below, with the eigenvalues (the eigenvectors are a little messy):

$$J = \frac{1}{5} \begin{bmatrix} -4 & -2 \\ -21 & -7 \end{bmatrix} \quad \lambda_{1,2} = -0.385, -1.814$$

This is a SINK. This implies that, for every non-zero set of initial populations, the populations over time will “peacefully coexist”, with populations remaining at the level at the sink.

The equations in the next problem are very similar, but lead to a much different conclusion:

2. Exercise 3, Section 9.4:

$$\begin{aligned}\dot{x} &= x(1.5 - 0.5x - y) \\ \dot{y} &= y(2 - y - 1.125x)\end{aligned}$$

Find the critical points, and at each compute the Jacobian matrix for classification. For extra practice, we should be able to compute the eigenvalues and eigenvectors as well. I will provide the plots here as well (you do not have to sketch them).

The equilibria are and the Jacobian matrix are (very similar to Exercise 1):

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 11/10 \\ 4/5 \end{bmatrix} \quad J = \begin{bmatrix} 1.5 - x - y & -x \\ -9/8y & 2 - 2y - 9/8x \end{bmatrix}$$

We isolate and analyze the solution close to each of the equilibria:

- At the origin, we have exactly the same situation as Exercise 1:

$$J = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_{1,2} = 1.5, 2 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- At (0, 2), the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} -1/2 & 0 \\ -9/4 & -2 \end{bmatrix} \quad \lambda_{1,2} = -2, -1/2 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

This is a SINK now. That means the interior saddle must have become a source- We shall see.

- At (3, 0), the local matrix is again given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} -3/2 & -3 \\ 0 & -11/8 \end{bmatrix} \quad \lambda_{1,2} = -3/2, -11/8 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -24 \\ 1 \end{bmatrix}$$

This is another SINK. Do you see what is about to happen?

- At (4/5, 11/10), the local matrix is again given below, with the eigenvalues (the eigenvectors are a little messy):

$$J = \begin{bmatrix} -0.4 & -0.8 \\ -1.24 & -1.1 \end{bmatrix} \quad \lambda_{1,2} = 0.305, -1.804$$

This is a SADDLE. This implies that, for every non-zero set of initial populations, one of the two populations will go extinct- a very big change from Exercise 1.

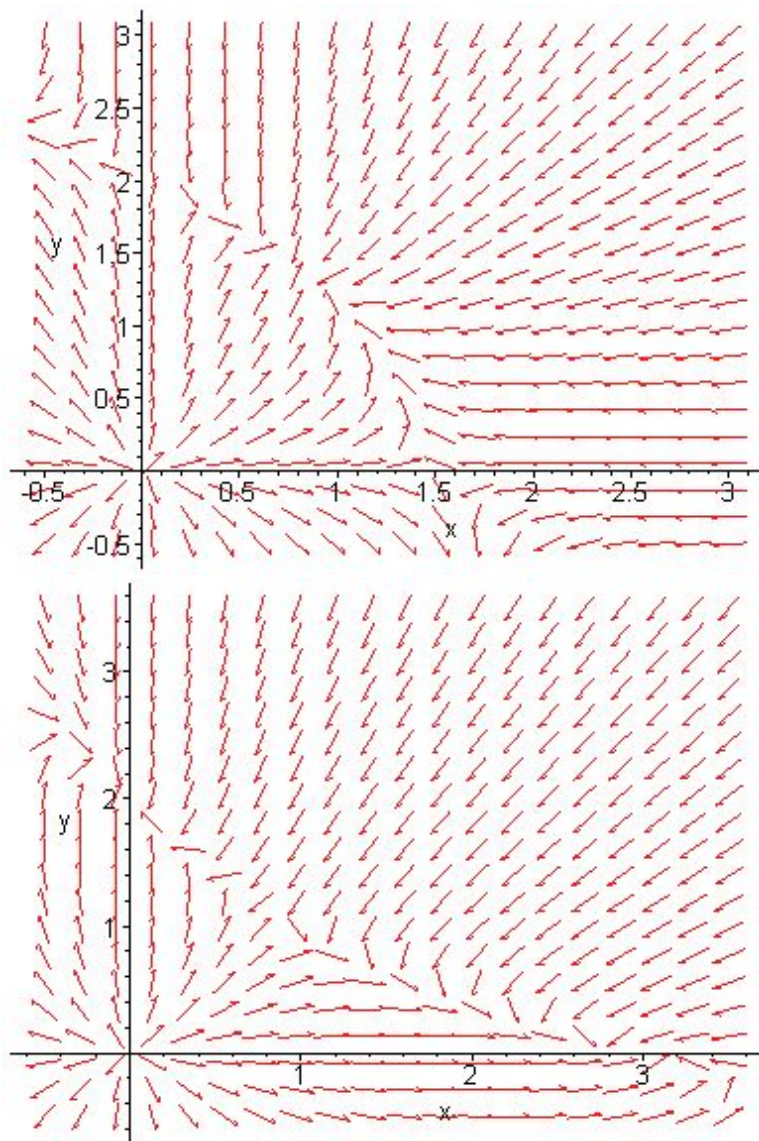


Figure 1: Phase planes for Exercises 1 and 3, Section 9.4. Notice that a small change in the differential equations led to completely different types of long term behavior.

For the next two problems, we consider only slightly different differential equations, but now we have one population hunting the other population for food. These are versions of the more general “Predator-Prey” equations:

3. Exercise 2, Section 9.5:

$$\begin{aligned}\dot{x} &= x(1 - 0.5y) \\ \dot{y} &= y(-0.25 + 0.5x)\end{aligned}$$

Find the critical points, and at each compute the Jacobian matrix for classification. For extra practice, we should be able to compute the eigenvalues and eigenvectors as well. I will provide the plots here as well (you do not have to sketch them).

The equilibria are and the Jacobian matrix are:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 2 \end{bmatrix}, \quad J = \begin{bmatrix} 1 - 0.5y & -0.5x \\ 0.5y & -0.25 + 0.5x \end{bmatrix}$$

We isolate and analyze the solution close to each of the equilibria:

- At the origin, the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix} \quad \lambda_{1,2} = 1, -1/4 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is a classic SADDLE. A set of initial conditions on the x -axis will lead to growth, and on the y -axis will lead to extinction.

- At $(1/2, 2)$, the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} 0 & -1/4 \\ 1 & 0 \end{bmatrix} \quad \lambda_1 = \frac{1}{2}i \quad \mathbf{v} = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

This is a CENTER. Typically, we cannot predict if the *nonlinear* system will still have a center at the equilibrium, but in this case it does (see the phase plane).

4. Exercise 3, Section 9.5:

$$\begin{aligned}\dot{x} &= x(1 - 0.5x - 0.5y) \\ \dot{y} &= y(-0.25 + 0.5x)\end{aligned}$$

This is a slight change from Exercise 2- We added an environmental threshold to the prey equation. This adds an extra equilibrium solution at $x = 2$, and we will see that the center becomes a spiral sink.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}, \quad J = \begin{bmatrix} 1 - x - 0.5y & -0.5x \\ 0.5y & -0.25 + 0.5x \end{bmatrix}$$

We isolate and analyze the solution close to each of the equilibria:

- At the origin, the local matrix is given below, with the eigenvalues and eigenvectors:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1/4 \end{bmatrix} \quad \lambda_{1,2} = 1, -1/4 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is again a SADDLE.

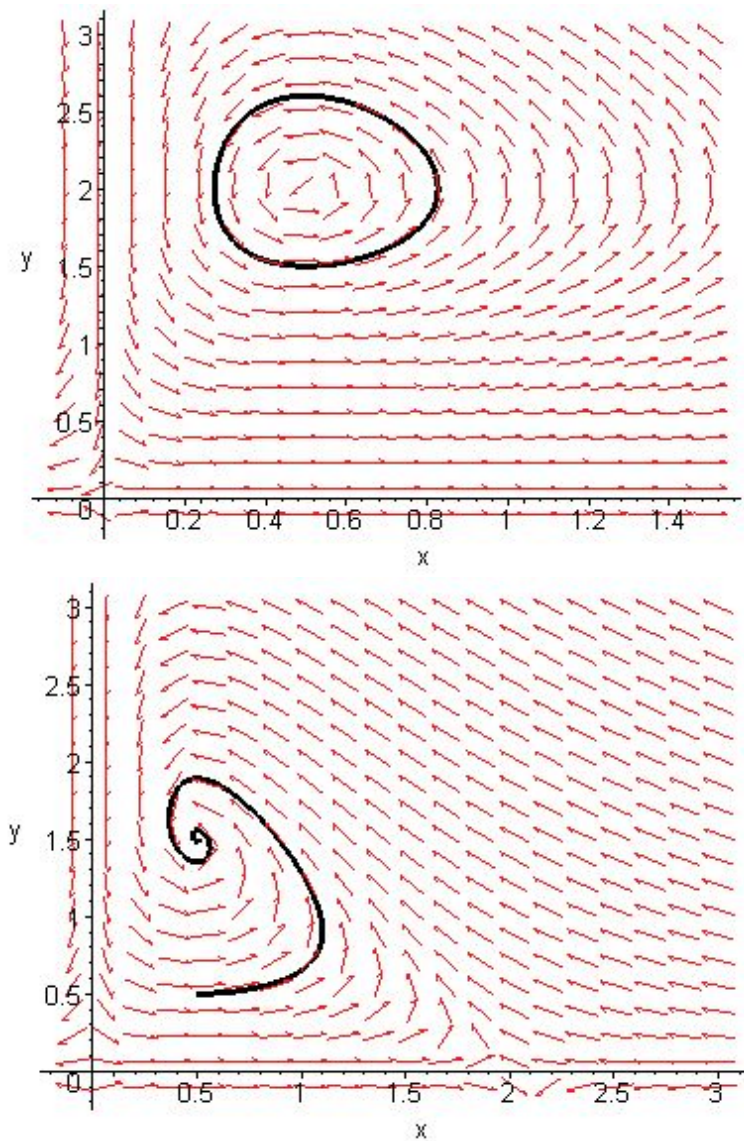


Figure 2: Figures for Exercises 2 and 3, Section 9.5 (Predator-Prey)

- At the new equilibrium $(2, 0)$ we might expect a saddle:

$$J = \begin{bmatrix} -1 & -1 \\ 0 & 3/4 \end{bmatrix} \quad \lambda_{1,2} = -1, 3/4 \quad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

This is a saddle, as we predicted. Now for the last equilibrium:

- At $(1/2, 3/2)$, we do indeed see a spiral sink:

$$J = \begin{bmatrix} -1/4 & -1/4 \\ 3/4 & 0 \end{bmatrix} \quad \lambda = -\frac{1}{8} + \frac{1}{8}i$$