Homework Solutions: 1.1-1.3

Section 1.1:

1. Problems 1, 3, 5

   In these problems, we want to compare and contrast the direction fields for the given (autonomous) differential equations of the form $y' = ay + b$. Once this is done, we want to be able to predict the direction field for the more general case.

   - Problem 1: $y' = 3 - 2y$. We should see that all solutions tend towards the equilibrium: $3 - 2y = 0$, or $y = 3/2$.
   - Problem 3: $y' = 3 + 2y$. In this case, the equilibrium changes to $y = -3/2$, and like Problem 2, all other solutions will tend towards either positive or negative infinity (predictable when the solution starts above or below $-3/2$, respectively).
   - Problem 5: $y' = 1 + 2y$ The equilibrium is again $y = -1/2$, except now the solutions move away from the equilibrium, going to $\pm\infty$ as $t \to \infty$ (again, that depends on the initial condition being above or below equilibrium).

2. Problem 7: If we want all solutions to tend towards $y = 3$, that will need to be the equilibrium. Furthermore, in the equation $y' = ay + b$, the value of a needs to be negative. There are lots of possibilities; here is one:

   $$y' = -y + 3$$

3. Problem 9: All solutions tend away from $y = 2$. In this case, the value of a in $y' = ay+b$ needs to be positive, and we can write something like:

   $$y' = y - 2$$

   Summary for Problems 1-9: For $y' = ay + b$, the equilibrium solution is where $y' = 0$, or where $ay + b = 0$. This gives:

   $$y = -b/a$$

   We can tell if the equilibrium is *attracting* (all solutions tend towards the equilibrium) or *repelling* (all solutions tend away from equilibrium) based on the sign of a. If $a > 0$, the equilibrium is repelling. If $a < 0$, the equilibrium is attracting.

4. Problem 14: Draw the direction field by hand. We will look at this problem more closely in Section 2.3.

5. Problems 15-20: Match the direction field to the DE: Pay particular attention to where the derivative is zero.

6. Problem 23 (See Problem 15, p. 17)

7. Problems 27-29: Were you able to sketch a graph by hand using a couple of isoclines? 27 was a little tricky because of the exponential, but hopefully you were able to do 29.

Section 1.2:

1. Problem 1(a,b). Very similar to the drawings from 1.1. In both cases, all solutions tend to the equilibrium solution.

2. Problem 3: $y' = -ay + b$

   (a) The solution is found by:

   $$y' = -a\left(y - \frac{b}{a}\right) \quad \Rightarrow \quad \frac{1}{y - b/a}\,dy = -a\,dt \quad \Rightarrow \quad \int \frac{1}{y - b/a}\,dy = \int -a\,dt \Rightarrow$$

   $$\ln|y - b/a| = -at + C \quad \Rightarrow \quad y - \frac{b}{a} = \mathrm{e}^{-at+C} = \mathrm{e}^{-at}\mathrm{e}^{C} = A\mathrm{e}^{-at}$$

   So that the solution is:

   $$y(t) = \frac{b}{a} + A\mathrm{e}^{-at}$$

   (b) Your graph in this case should have a horizontal solution (the equilibrium solution) at $y = b/a$. The slopes above the equilibrium should go down, the slope below should point up.

   (c) Describe how the solution changes under each of the following conditions:

      i. a increases: This makes the solutions go to equilibrium faster than before (the slopes are made more steep). Changing a and leaving b fixed also makes the equilibrium get smaller.

      ii. b increases: Does not change the rate at which the solutions go to the equilibrium, but does change the equilibrium (if b increases, the equilibrium also increases).

      iii. Both a, b increase, but the ratio b/a stays fixed. This will change the rate at which solutions go to the equilibrium, which stays fixed.

3. Problem 4: The only difference between problems 3 and 4 is that we've multiplied by negative 1. The equilibrium solution is:

   $$y' = 0 \quad \Rightarrow \quad 0 = ay - b \quad \Rightarrow \quad y = b/a$$

It is hard to read the difference between small case y and capital Y- We'll use W instead. We are finding a differential equation for $W = y - b/a$. We see that:

$$W' = y' = ay - b = a\left(W + \frac{b}{a}\right) - b = aW$$

so the new DE: $W' = aW$.

4. Problem 5: Undetermined Coefficients.

   In this problem, we want to compare the solutions to:

   $$y' = ay \qquad \text{versus} \qquad y' = ay - b$$

   The solution to the first equation is: $y(t) = A\mathrm{e}^{at}$. To find the solution to the second, we **assume** that the solution is of the form:

   $$y(t) = A\mathrm{e}^{at} + k$$

   for some unknown k. Our problem is now to find k, which we do by substituting our guess into the differential equation.

   The left hand side of the D.E. is just y', so if $y = A\mathrm{e}^{at} + k$, then $y' = aA\mathrm{e}^{at}$.

   The right hand side of the D.E. is $ay-b$, so if $y = A\mathrm{e}^{at}+k$, this becomes $a(A\mathrm{e}^{at}+k)-b$.

   Now equate the left and right hand sides, and solve for k:

   $$aA\mathrm{e}^{at} = aA\mathrm{e}^{at} + ak - b \quad \Rightarrow \quad 0 = ak - b \quad \Rightarrow \quad k = b/a$$

   Therefore, the overall solution is (what we had before):

   $$y(t) = A\mathrm{e}^{at} + \frac{b}{a}$$

5. Problem 6: Solve $y' = -ay + b$ using Problem 5:

   First, the solution to the simpler DE: $y' = -ay$ is $y(t) = A\mathrm{e}^{-at}$. We guess that the solution to $y' = -ay + b$ will be of the form $y = A\mathrm{e}^{-at} + k$. To find k, substitute into the DE:

   $$y' = -aA\mathrm{e}^{-at} \qquad -ay + b = -aA\mathrm{e}^{-at} - ak + b$$

   These have to be the same so: $0 = -ak + b$, or $k = b/a$. Therefore the solution is:

   $$y(t) = A\mathrm{e}^{-at} + \frac{b}{a}$$

6. Problem 7 (Field Mice): $p' = \frac{1}{2}p - 450$

(a) From the previous two problems (or with the technique from the Chapter), we can write down the solution:

$$\frac{dp}{dt} = \frac{1}{2}(p - 900) \qquad \frac{dp}{p - 900} = \frac{1}{2}\,dt$$

And integrate both sides:

$$\ln|p - 900| = \frac{1}{2}t + C \quad \Rightarrow \quad p(t) = A\mathrm{e}^{(1/2)t} + 900$$

Now, if $p(0) = 850$, we can get the particular solution (solve for A):

$$p(0) = A + 900 = 850 \quad \Rightarrow \quad A = -50$$

Therefore, $p(t) = -50\mathrm{e}^{(1/2)t} + 900$. To say that the population became extinct means that the population is zero. Set $p(t) = 0$ and solve for t:

$$-50\mathrm{e}^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad \mathrm{e}^{(1/2)t} = 18 \quad \Rightarrow t = 2\ln(18) \approx 5.78$$

(b) Similarly, if $p(0) = p_0$, with $0 < p_0 < 900$,

$$p_0 = A + 900 \Rightarrow A = p_0 - 900$$

and:

$$(p_0 - 900)\mathrm{e}^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad \mathrm{e}^{(1/2)t} = \frac{-900}{p_0 - 900} = \frac{900}{900 - p_0}$$

(I wrote the last fraction like that so it would be clear that this is a positive number before we take the log of both sides)

Therefore, our conclusion is: Given $p' = \frac{1}{2}p - 450$, $p(0) = p_0$, where $0 < p_0 < 900$, then the time at which extinction occurs is:

$$t = 2\ln\left(\frac{900}{900 - p_0}\right)$$

(c) Find the initial population if the population becomes extinct in one year. Note that t is measured in months, so that would mean that we want to solve our general equation for p_0 if $p(12) = 0$. We can use our last result:

$$12 = 2\ln\left(\frac{900}{900 - p_0}\right)$$

Solve for p_0:

$$\frac{900}{900 - p_0} = \mathrm{e}^6 \quad \Rightarrow \quad 900\mathrm{e}^{-6} = 900 - p_0 \quad \Rightarrow \quad p_0 = 900 - 900\mathrm{e}^{-6}$$

7. Problem 8: More mice! The population at time t is $p(t)$, and we have the exponential growth model:

$$\frac{dp}{dt} = rp$$

The solution is $p(t) = P_0 \mathrm{e}^{rt}$, where P_0 is the initial population. If that population doubles in 30 days, **and t is measured in days**, we can write:

$$2P_0 = P_0 \mathrm{e}^{30r} \quad \Rightarrow \quad r = \ln(2)/30$$

If the population doubles in N days, we see that $r = \ln(2)/N$.

8. Problem 15 (Newton's Law of Cooling):

We are given:

$$\frac{du}{dt} = -k(u - T), \qquad u(0) = u_0$$

We can solve this either directly or using the techniques from this HW. Directly,

$$\frac{1}{u-T}\,dt = -k\,dt \quad \Rightarrow \quad \int \frac{1}{u-T}\,du = \int -k\,dt \quad \Rightarrow \quad \ln|u - T| = -kt + C$$

Now solve for $u(t)$:

$$u - T = \mathrm{e}^{-kt+c} = \mathrm{e}^{-kt}\mathrm{e}^{c} = A\mathrm{e}^{-kt}$$

Also, find A in terms of the initial condition, $u(0) = u_0$:

$$u(0) = A + T = u + 0 \quad \Rightarrow \quad A = u_0 - T$$

In conclusion, the temperature at any time t:

$$u(t) = (u_0 - T)\mathrm{e}^{-kt} + T$$

Part (b) is a little trickier, in that we need to properly translate the statement:

> Let τ be the time at which the initial temperature difference, $u_0 - T$ has been reduced by half. Find the relation between k and τ

If $u(t)$ is the actual temperature at time t, then $u(t) - T$ is the temperature *difference* at any time t between $u(t)$ and T. The statement is then translated to read:

$$u(\tau) - T = \frac{1}{2}(u_0 - T)$$

Now substitute and solve for k:

$$(u_0 - T)\mathrm{e}^{-k\tau} + T - T = \frac{1}{2}(u_0 - T)$$

So that:

$$\mathrm{e}^{-k\tau} = \frac{1}{2} \quad \Rightarrow \quad -k\tau = \ln(1/2) = -\ln(2) \quad \Rightarrow \quad k = \ln(2)/\tau$$

Section 1.3

1. Problem 1: Order is 2, and it is linear (divide by the leading t^2)
2. Problem 3: Order is 4, and it is linear.
3. Problem 5: Order is 2, and nonlinear (because of $\sin(t+y)$ term).
4. Problem 7: Do you know the definition of $\cosh(t)$? See our class website before doing this problem- There are practice problems there). You might use the definition directly, or from the practice sheet, see that:

$$\frac{d}{dx}(\cosh(x)) = \sinh(x) \qquad \frac{d}{dx}(\sinh(x)) = \cosh(x)$$

   Now, to solve problem 7, we want to verify that either $y(t) = \mathrm{e}^t$ or $y(t) = \cosh(t)$ satisfies the differential equation: $y'' - y = 0$.

   If $y(t) = \mathrm{e}^t$, then $y'(t) = \mathrm{e}^t$, and $y''(t) = \mathrm{e}^t$, so

$$y'' - y = \mathrm{e}^t - \mathrm{e}^t = 0$$

   If $y(t) = \cosh(t)$, then $y' = \sinh(t)$ and $y''(t) = \cosh(t)$, so again,

$$y'' - y = \cosh(t) - \cosh(t) = 0$$

5. Problem 9: Show that $y(t) = 3t + t^2$ satisfies the ODE: $ty' - y = t^2$.

   First compute the derivative, then substitute into the expression:

$$y' = 3 + 2t$$

   so that:

$$ty' - y = t(3+2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$$

6. Problem 14: Show that the function

$$y(t) = \mathrm{e}^{t^2} \int_0^t \mathrm{e}^{-s^2}\, ds + \mathrm{e}^{t^2}$$

   solves: $y' - 2ty = 1$.

   To show this directly, we need to recall how to differentiate a function like:

$$g(t) = \int_0^t f(s)\, ds$$

   From the Fundamental Theorem of Calculus, $g'(t) = f(t)$.

Therefore, if $y(t)$ is as given above, the derivative is found by using the product rule:

$$y' = \left(2te^{t^2}\right) \cdot \int_0^t e^{-s^2}\,ds + e^{t^2}e^{-t^2} + 2te^{t^2}$$

If we simplify a bit, and subtract:

$$y' = 2te^{t^2} \int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$$

$$-2ty = -2t\left(e^{t^2} \int_0^t e^{-s^2}\,ds + e^{t^2}\right)$$

We see that the only remaining term is 1.

(NOTE: In Section 2.1, we'll see where this strange integral is coming from)

7. Problem 15: We did something similar in class: If $y = e^{rt}$, substitute it into the differential equation-

$$y' + 2y = 0 \quad \Rightarrow \quad re^{rt} + 2e^{rt} = 0$$

Now solve for r:

$$(r+2)e^{rt} = 0 \quad \Rightarrow \quad r + 2 = 0 \quad \Rightarrow \quad r = -2$$

Note that $e^{rt} = 0$ has no solution.

Conclusion: $y(t) = e^{-2t}$.

Side Remark: We solved this in Section 1.2 by doing this:

$$y' = -2y \quad \Rightarrow \quad \frac{1}{y}\,dy = -2\,dt \quad \Rightarrow \quad \int \frac{1}{y}\,dy = -2\int\,dt$$

so that:

$$\ln|y| = -2t + c \quad \Rightarrow \quad y(t) = Ae^{-2t}$$

8. Problem 17: Same setup as Problem 15: If $y(t) = e^{rt}$,

$$y'(t) = re^{rt} \qquad y''(t) = r^2e^{rt}$$

Substitute these into the DE: $y'' - y' - 6y = 0$ and solve for r:

$$r^2e^{rt} + re^{rt} - 6e^{rt} = 0 \quad \Rightarrow \quad e^{rt}\left(r^2 + r - 6\right) = 0$$

Again, $e^{rt} = 0$ has no solution, so just solve:

$$r^2 + r - 6 = 0 \qquad (r+3)(r-2) = 0 \qquad r = -3, 2$$

Either $y = e^{-3t}$ or $y = e^{2t}$ will solve the DE.

9. Problem 19: In this case, assume $y = t^r$, so $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substitute these into the DE:

$$t^2 y'' + 4ty' + 2y = 0 \quad \Rightarrow \quad t^2 \cdot r(r-1)t^{r-2} + 4t \cdot rt^{r-1} + 2t^r = 0$$

Simplify and factor out t^r:

$$t^r(r(r-1) + 4r + 2) = 0$$

This equation must be true for ALL $t > 0$ (given in the problem), so $t^r = 0$ does not give a solution. Solve for r:

$$r^2 - r + 4r + 2 = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+1)(r+2) = 0$$

Therefore, $y(t) = \frac{1}{t}$ and $y(t) = \frac{1}{t^2}$ solve the differential equation.

10. Problem 21: The order is 2, linear.

11. Problem 25: Show that each of these:

$$u(x,y) = \cos(x)\cosh(y) \qquad u(x,y) = \ln(x^2 + y^2)$$

solve the Partial Differential Equation (PDE):

$$u_{xx} + u_{yy} = 0$$

If $u(x,y) = \cos(x)\cosh(y)$, then

$$u_x = -\sin(x)\cosh(y) \qquad u_{xx} = -\cos(x)\cosh(y)$$

Similarly,

$$u_y = \cos(x)\sinh(y) \qquad u_{yy} = \cos(x)\cosh(y)$$

And if we add u_{xx} to u_{yy}, we get zero.

If $u(x,y) = \ln(x^2 + y^2)$, then:

$$u_x = \frac{2x}{x^2 + y^2} \qquad u_{xx} = \frac{-2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Similarly,

$$u_y = \frac{2y}{x^2 + y^2} \qquad u_{yy} = \frac{-2(y^2 - x^2)}{(x^2 + y^2)^2}$$

And again we see that if we add u_{xx} and u_{yy}, we get zero.