Homework Solutions: 2.1-2.2, plus Substitutions

Section 2.1

I have not included any drawings/direction fields. We can see them using Maple or by hand, so we'll be focusing on getting the analytic solutions here:

1. Problem 1: Solve the DE using the Method of Integrating Factor (for linear differential equations):

$$y' + 3y = t + \mathrm{e}^{-2t} \quad \Rightarrow \quad \mathrm{e}^{3t}\left(y' + 3y\right) = \mathrm{e}^{3t}\left(t + \mathrm{e}^{-2t}\right) \quad \Rightarrow \quad \left(\mathrm{e}^{3t}y(t)\right)' = t\mathrm{e}^{3t} + \mathrm{e}^{t}$$

   Integrate both sides *Hint:* We need to use "integration by parts" to integrate $t\mathrm{e}^{3t}$. Using a table as in class:

$$\begin{array}{|c|c|c|} + & t & \mathrm{e}^{3t} \\ - & 1 & (1/3)\mathrm{e}^{3t} \\ + & 0 & (1/9)\mathrm{e}^{3t} \end{array} \quad \Rightarrow \quad \int t\mathrm{e}^{3t}\,dt = \frac{1}{3}\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t}$$

   Putting it all together,

$$\mathrm{e}^{3t}y(t) = \frac{1}{3}t\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t} + \mathrm{e}^{t} + C$$

   so that

$$y(t) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{\mathrm{e}^{-2t}} + \frac{C}{\mathrm{e}^{3t}}$$

   Notice that the last two terms go to zero as $t \to \infty$, so we see that $y(t)$ does approach a line:

$$\frac{1}{3}t - \frac{1}{9}$$

   as $t \to \infty$.

2. Problem 3: Very similar situation to Problem 1. Let's go ahead and solve:

$$y' + y = t\mathrm{e}^{-t} + 1$$

   Multiply both sides by $\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{t}$:

$$\mathrm{e}^{t}\left(y' + y\right) = t + \mathrm{e}^{t} \quad \Rightarrow \quad \left(\mathrm{e}^{t}y(t)\right)' = t + \mathrm{e}^{t}$$

   Integrate both sides:

$$\mathrm{e}^{t}y(t) = \frac{1}{2}t^2 + \mathrm{e}^{t} + C \quad \Rightarrow \quad y(t) = \frac{1}{2}t^2\mathrm{e}^{-t} + 1 + C\mathrm{e}^{-t}$$

   This could be written as:

$$y(t) = 1 + \frac{t^2}{2\mathrm{e}^{t}} + \frac{C}{\mathrm{e}^{t}}$$

   so that it is clear that, as $t \to \infty$, $y(t) \to 1$.

3. Problem 5:
$$y' - 2y = 3\mathrm{e}^t \quad \Rightarrow \quad \left(\mathrm{e}^{-2t}y\right)' = \mathrm{e}^{-t}$$
Finishing, we get:
$$y(t) = -3\mathrm{e}^t + C\mathrm{e}^{2t}$$
(All solutions tend to $\pm\infty$)

4. Problem 7:
$$y' + 2ty = 2t\mathrm{e}^{t^2} \quad \Rightarrow \quad \left(\mathrm{e}^{t^2}y\right)' = 2t$$
so that $y(t) = (t^2 + C)\mathrm{e}^{-t^2}$, and all solutions tend to zero as $t \to \infty$.

5. Problem 13: (You'll need to integrate by parts!)
$$y' - y = 2t\mathrm{e}^{2t} \qquad \mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{-t}$$
$$y(t) = \mathrm{e}^{2t}(2t - 2) + 3\mathrm{e}^t$$

6. Problem 15:
$$ty' + 2y = t^2 - t + 1$$
Be sure to put in standard form before solving:
$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \qquad \mathrm{e}^{\int p(t)\,dt} = t^2$$
and
$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

7. Problem 16: In this problem, the integrating factor is again t^2:
$$y' + \frac{2}{t} \cdot y = \frac{\cos(t)}{t^2} \qquad \Rightarrow \qquad y(t) = \frac{\sin(t)}{t^2}$$

8. Problem 30:
$$y' - y = 1 + 3\sin(t) \qquad y(0) = y_0$$
This is very similar to Problem 29. Note that:
$$\int \mathrm{e}^{-t}\sin(t)\,dt = -\frac{1}{2}\mathrm{e}^{-t}\left(\cos(t) + \sin(t)\right)$$
Therefore, the general solution is (details left out):
$$y(t) = -1 - \frac{3}{2}\left(\cos(t) + \sin(t)\right) + \left(\frac{5}{2} + y_0\right)\mathrm{e}^t$$
To keep the solution finite (or bounded) as $t \to \infty$, we must find y_0 so that the exponential term drops out- This means that $y_0 = -5/2$.

9. Problem 35: There are many ways of constructing such a differential equation- It's easiest to start with a desired solution. We'll again show two possibilities:

   - If we would like $y(t) = 3 - t + C\mathrm{e}^{-3t}$, then $y' = -1 - 3C\mathrm{e}^{-3t}$, and:
   $$y' + 3y = 8 - 3t$$
   - If we would like $y(t) = 3 - t + \frac{C}{t}$, then $y' = -1 - C/t^2$, and we see that:
   $$ty' + y = 3 - 2t$$

10. Problem 36: Similar to 35, let's try a solution then construct a DE. In this case, if $y = 2t - 5 + C\mathrm{e}^{-t}$, then $y' = 2 - C\mathrm{e}^{-t}$, so that:
$$y' + y = 2t - 3$$

2.2

1. Problem 1: Give the general solution: $y' = x^2/y$
$$y\,dy = x^2\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

2. Problem 3: Give the general solution to $y' + y^2\sin(x) = 0$.

   First write in standard form:
$$\frac{dy}{dx} = -y^2\sin(x) \quad \Rightarrow \quad -\frac{1}{y^2}\,dy = \sin(x)\,dx$$
   Before going any further, notice that we have divided by y, so we need to say that this is value as long as $y(x) \neq 0$. In fact, we see that the function $y(x) = 0$ IS a possible solution.

   With that restriction in mind, we proceed by integrating both sides to get:
$$\frac{1}{y} = -\cos(x) + C \quad \Rightarrow \quad y = \frac{1}{C - \cos(x)}$$

3. Problem 5: This one is good for a little extra practice in integrating. Recall that:
$$\int \cos^2(x)\,dx = \frac{1}{2}\int (1 + \cos(2x))\;dx = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right)$$
   Given that,
$$y' = \cos^2(x)\cos^2(2y) \quad \Rightarrow \quad \sec^2(2y)\,dy = \cos^2(x)\,dx \quad \Rightarrow$$

$$\frac{1}{2}\tan(2y) = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$$

It would be fine to leave it in this form. Of course, this solution is only valid when $\cos(2y) \neq 0$. The constant solutions

$$\cos(2y) = 0$$

would also be solutions (we'll focus on these types of solutions later). Solving for y, we get the answer in the text.

4. Problem 7: Give the general solution:

$$\frac{dy}{dx} = \frac{x - \mathrm{e}^{-x}}{y + \mathrm{e}^{y}}$$

First, note that dy/dx exists as long as $y \neq \mathrm{e}^y$. With that requirement, we can proceed:

$$(y + \mathrm{e}^y)\ dy = \left(x + \mathrm{e}^{-x}\right)\ dx$$

Integrating, we get:

$$\frac{1}{2}y^2 + \mathrm{e}^y = \frac{1}{2}x^2 - \mathrm{e}^{-x} + C$$

In this case, we cannot algebraically isolate y, so we'll leave our answer in this form (we could multiply by two).

5. Problem 9: Let $y' = (1 - 2x)y^2, \qquad y(0) = -1/6$.

First, we find the solution. Before we divide by y, we should make the note that $y \neq 0$. We also see that $y(x) = 0$ is a possible solution (although NOT a solution that satisfies the initial condition).

Now solve:

$$\int y^{-2}\, dy = \int (1 - 2x)\, dx \quad \Rightarrow \quad -y^{-1} = x - x^2 + C$$

Solve for the initial value:

$$6 = 0 + C \Rightarrow C = 6$$

The solution is (solve for y):

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x - 3)(x + 2)}$$

The solution is valid only on $-2 < x < 3$, and we could plot this by hand (also see the Maple worksheet).

6. Problem 11: $x\,dx + y\mathrm{e}^{-x} dy = 0, \qquad y(0) = 1$

   To solve, first get into a standard form, multiplying by e^x, and integrate (integration by parts for the right hand side):

$$\int y\,dy = -\int x\mathrm{e}^x\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -x\mathrm{e}^x + \mathrm{e}^x + C$$

   We could solve for the constant before isolating y:

$$\frac{1}{2} = 0 + 1 + C \quad C = -\frac{1}{2}$$

   Now solve for y:

$$y^2 = 2\mathrm{e}^x(x-1) - \frac{1}{2}$$

   and take the positive root, since $y(0) = +1$.

$$y = \sqrt{2\mathrm{e}^x(1-x) - 1}$$

   The solution exists as long as:

$$2\mathrm{e}^x(1-x) - 1 \geq 0$$

   We would have to use Maple to solve where this is equal to zero- Given software, we could see that $-1.678 \leq x \leq 0.768$.

7. Problem 16:

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3} \qquad y(0) = -\frac{1}{\sqrt{2}}$$

   First, we notice that $y \neq 0$. Now separate the variables and integrate:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$

   This might be a good time to solve for C: $C = 1/4$, so:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{4}$$

   The right side of the equation seems to be a nice form. Try some algebra to simplify it:

$$\frac{1}{4}\left(x^4 + 2x^2 + 1\right) = \frac{1}{4}(x^2+1)^2$$

   Now we can write the solution:

$$y^4 = \frac{1}{4}(x^2+1)^2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt{2}}\sqrt{x^2+1}$$

   This solution exists for all x (it is the bottom half of a hyperbola- see the Maple plot).

8. Problem 20: $y^2\sqrt{1-x^2}dy = \sin^1(x)\,dx$ with $y(0) = 1$.

   To put into standard form, we'll be dividing so that $x \neq \pm 1$. In that case,

$$\int y^2\,dy = \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

   The right side of the equation is all set up for a u, du substitution, with $u = \sin^{-1}(x)$, $du = 1/\sqrt{x^2-1}\,dx$:

$$\frac{1}{3}y^3 = \frac{1}{2}(\arcsin(x))^2 + C$$

   Solve for C, $\frac{1}{3} = 0 + C$ so that:

$$\frac{1}{3}y^3 = \frac{1}{2}\arcsin^2(x) + \frac{1}{3}$$

   Now,

$$y(x) = \sqrt[3]{\frac{3}{2}\arcsin^2(x) + 1}$$

   The domain of the inverse sine is: $-1 \leq x \leq 1$. However, we needed to exclude the endpoints. Therefore, the domain is:

$$-1 < x < 1$$

Substitution Methods

See the PDF file that is linked online - It shows the direction fields so that you can see the "zoom invariance" we were talking about in class.

- pg. 50, 31: $y' = (x^2 + xy + y^2)/x^2$ Divide it out:

$$y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

  Our substitution will be $v = y/x$, or $y = xv$ so that $y' = v + xv'$:

$$v + xv' = 1 + v + v^2 \quad \Rightarrow \quad xv' = 1 + v^2$$

  This is now separable:

$$\frac{1}{1+v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow \quad \tan^{-1}(v) = \ln|x| + C \quad \Rightarrow$$

  Remember to back-substitute. You can typically leave your answer in implicit form:

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C$$

- p. 50, 33: Divide numerator and denominator by x and substitute $v = y/x$, $y = xv$ and $y' = v + xv'$ to translate the differential equation to:

$$v + xv' = \frac{4v-3}{2-v} \qquad \Rightarrow \qquad xv' = \frac{4v-3}{2-v} - \frac{v(2-v)}{2-v} = \frac{v^2+2v-3}{-(v-2)}$$

Separation of variables gives:

$$\int \frac{-(v-2)}{v^2+2v-3}\, dv = \int \frac{1}{x}\, dx$$

To integrate the left side, we use partial fractions:

$$\frac{-v+2}{v^2+2v-3} = \frac{A}{v+3} + \frac{B}{v-1} = \frac{-5}{4}\frac{1}{v+3} + \frac{1}{4}\cdot\frac{1}{v-1}$$

Now integrate through, and multiply by 4:

$$-\frac{5}{4}\ln|v+3| + \frac{1}{4}\ln|v-1| = \ln|x| + C$$

Back-substitute. We can simplify a bit:

$$\ln\left|\frac{v-1}{(v+3)^4}\right| = \ln(x^4) + C \qquad \Rightarrow \qquad \frac{v-1}{(v+3)^5} = Ax^4 \qquad \Rightarrow \qquad \left|\frac{y}{x} - 1\right| = Ax^4\left|\frac{y}{x}+3\right|^5$$

Multiply both sides by $|x|$ to get the answer in the text,

$$|y - x| = A|y + 3x|^5$$

Of course, this was only valid if $v - 1 \neq 0$ and $v + 3 \neq 0$. Notice that in these cases, we do have solutions:

$$y = x \qquad y = -3x$$

(Try substituting them back in!)

- p. 50, 35: $y' = (x+3y)/(x-y)$ Substitute as usual, then simplify:

$$v + xv' = \frac{1+3v}{1-v} \qquad \Rightarrow \qquad xv' = \frac{(v+1)^2}{-v+1}$$

Separate variables, and integrate (You'll need partial fractions, shown below):

$$\int \frac{-v+1}{(v+1)^2}\, dv = \ln|x| + C$$

where

$$\frac{-v+1}{(v+1)^2} = \frac{A}{v+1} + \frac{B}{(v+1)^2} = -\frac{1}{v+1} + \frac{2}{(v+1)^2}$$

And this antiderivative is $-\ln|v+1| - \frac{2}{v+1}$. Put it together to get:

$$-\ln|v+1| - \frac{2}{v+1} = \ln|x| + C$$

Back substitute for v and simplify:

$$\ln\left|\frac{x}{y+x}\right| - \frac{2x}{y+x} = \ln|x| + C \quad \Rightarrow \quad \ln|y+x| + \frac{2x}{y+x} = C_2$$

(NOTE: We do want to simplify some; I'm showing how to get the form in the back of the book)

- p. 77, 28: For the Bernoulli equations, we want to divide everything by the highest power of y. In this case (We also divide by t^2 to get standard form):

  $$\frac{y'}{y^3} + \frac{2}{t}\frac{1}{y^2} = \frac{1}{t^2}$$

  Now if $v = 1/y^2$, then

  $$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{y'}{y^3} = -\frac{1}{2}v'$$

  Substitute back in:

  $$-\frac{1}{2}v' + \frac{2}{t}v = \frac{1}{t^2}$$

  Now put this into a standard linear form, then get the integrating factor:

  $$v' - \frac{4}{t}v = -\frac{2}{t^2} \quad \Rightarrow \quad \mathrm{e}^{\int p(t)\,dt} = t^{-4}$$

  Therefore,

  $$\left(\frac{v}{t^4}\right)' = -\frac{2}{t^6} \quad \Rightarrow \quad v = \frac{2}{5t^5} + Ct^4$$

  Back substitute and make the right side of the equation into a single fraction for later:

  $$\frac{1}{y^2} = \frac{2 + C_2 t^5}{5t}$$

  which is where the answer in the text comes from.

- p. 77, 29: Similarly,

  $$y' = ry - ky^2 \quad \Rightarrow \quad y' - ry = -ky^2 \quad \Rightarrow \quad \frac{y'}{y^2} - r\frac{1}{y} = -k$$

From which we get

$$v = \frac{1}{y} \quad \Rightarrow \quad \frac{y'}{y^2} = -v'$$

so that

$$-v' - rv = -k \quad \Rightarrow \quad v' + rv = k \quad \Rightarrow \quad (v\mathrm{e}^{rt})' = k\mathrm{e}^{rt}$$

From which we get:

$$v = \frac{k}{r} + C\mathrm{e}^{-rt}$$

Back substitute $v = 1/y$ to get the answer in the text.