Sample Questions, Exam 1

Math 244 Spring 2007

Remember, on the exam you may use a calculator, but NOT one that can perform symbolic manipulation (remembering derivative and integral formulas are a part of the course).

These questions are presented to give you an idea of the variety and style of question that will be on the exam. You should also understand the homework problems and quizzes.

1 Short Answer/True or False

1. True or False, and explain: If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   False: (a) Equilibrium solutions are only defined for *autonomous* differential equations, (b) This is an isocline for a slope of zero, and (c) $y = -2t$ is not a solution.

2. State the Existence and Uniqueness Theorem for linear first order initial value problems (IVPs).

   Let $y' + p(t)y = g(t)$ with $y(t_0) = y_0$.

   If p, g are continuous on an open interval I containing t_0, then a unique solution exists to the IVP. In addition, the solution is valid on I.

   (Note: The interval I is a single (connected) interval, not two or more intervals).

3. State the general Existence and Uniqueness Theorem for first order initial value problems (IVPs).

   Let $y' = f(t, y)$ with $y(t_0) = y_0$.

   If f is continuous on an open rectangle containing (t_0, y_0), then a solution exists.

   If $\partial f/\partial y$ is continuous on an open rectangle containing (t_0, y_0), then the solution is unique.

   In the general case, we cannot predict ahead of time on what interval the solution will be valid- We have to solve the IVP.

4. Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them.

   It is, but only if f is not continuous. If f is continuous (which is a normal assumption on f), then it is not possible (draw a picture in the phase plane and you'll see why).

5. What is a *linear* first order differential equation.

   A linear first order DE is any DE that can be expressed as:

   $$y' + p(t)y = g(t)$$

6. Let $y' = \sin(y)$. It is possible that the solution can oscillate (or be periodic).

   False. In the direction field, the slopes cannot depend on t- That is, all of the slopes at a given y value are all the same. This means that solutions are monotonically increasing, monotonically decreasing, or are constant (equilibria). No solution can oscillate.

7. What is an n^{th} order differential equation?

   The order of a differential equation refers to the integer of the highest derivative. An n^{th} order DE would have an n^{th} derivative as the highest derivative.

8. What's the difference between:

   "The domain of $y(t)$" and "The time interval for which $y(t)$ is a solution to the DE"?

   When we talk about the time interval on which a solution is valid, the interval must be a single (connected) interval. A domain can be any combination of points, intervals, etc.

9. Let $\frac{dy}{dt} = 1 + y^2$. Then the solution will be valid for all t.

   False. Solving the DE:

$$\int \frac{1}{1+y^2}\,dy = \int dt \quad \Rightarrow \quad \tan^{-1}(y) = t + C \quad \Rightarrow \quad y = \tan(t+c)$$

   The tangent function has vertical asymptotes at $t + c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, so there will be a strip of time of length π on which the solution will be valid (but no more).

10. We said earlier that, under certain circumstances, solution curves to $y' = f(t, y)$ cannot cross (or be tangent) in the direction field. What are the conditions?

    The conditions are those stated in the existence and uniqueness theorem (for uniqueness in particular). If $\partial f/\partial y$ is continuous, solutions cannot cross (or be tangent) in the direction field.

11. Problems 15-20 on page 8 (these are graphical questions- Be sure you look these over!).

2 Solve:

Give the general solution if there is no initial value. Before giving the solution, state what kind of differential equation it is (linear, separable, exact)- Multiple classes are possible, just give the one you will use.

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

   Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x + y)\,dx - (x - y)\,dy = 0$. Hint: Let $v = y/x$.

   Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)} = \frac{1+v}{1-v}$$

   With the substitution $xv = y$, we get the substitution for dy/dx:

$$v + xv' = y'$$

   So that the DE becomes:

$$v + xv' = \frac{1+v}{1-v} \quad \Rightarrow \quad xv' = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1+v} = \frac{1+v^2}{1-v}$$

   The equation is now separable:

$$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln|x| + C$$

   Therefore,

$$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

   Lastly, back-substitute $v = y/x$.

3. $\dfrac{dy}{dx} = \dfrac{2x+y}{3+3y^2-x} \qquad y(0) = 0.$

   This is exact. The solution is, with $y(0) = 0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$

   This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0) = 2$

   This is linear and separable. $y(t) = \frac{2}{3}\sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. State the interval on which the solution is valid.

   This is linear and separable. The solution is $y(t) = 200\mathrm{e}^{(1/2)t}$

7. $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$

   This is linear (but not separable). $y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{(1/2)t}$

8. $y' = \frac{1}{2}y(3 - y)$

   Autonomous (and separable). Integrate using partial fractions:

   $$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int dt$$

   Simplify your answer for y by dividing numerator and denominator appropriately to get:

   $$y(t) = \frac{3}{(1/A)\mathrm{e}^{-(3/2)t} + 1}$$

9. $\sin(2t)\ dt + \cos(3y)\ dy = 0$

   Separable (and/or exact): $\frac{-1}{2}\cos(2t) + \frac{1}{3}\sin(3y) = C$

10. $y' = xy^2$

    Separable: $y = \dfrac{1}{-(1/2)x^2 - C}$

11. $\dfrac{dy}{dt} = \mathrm{e}^{t+y}$

    Separable: $y' = \mathrm{e}^t\mathrm{e}^y$, so:

    $$\int \mathrm{e}^{-y}\,dy = \int \mathrm{e}^t\,dt$$

    and $-\mathrm{e}^{-y} = \mathrm{e}^t + C$

12. $\dfrac{dy}{dx} + y = \dfrac{1}{1 + \mathrm{e}^x}$

    Linear: $y' + y = 1/(1 + \mathrm{e}^x)$, and the I.F. is e^x. Therefore,

    $$(\mathrm{e}^x y) = \int \frac{\mathrm{e}^x}{1 + \mathrm{e}^x}\,dx$$

    To integrate, use u, du substitution. The solution is then:

    $$y = \frac{\ln(1 + \mathrm{e}^x) + C}{\mathrm{e}^x}$$

13. $(t^2y + ty - y)\,dt + (t^2y - 2t^2)\,dy = 0$

    Does not seem to be exact. Try separating variables:

    $$\frac{dy}{dt} = \frac{-y(t^2 + t + 1)}{(y - 2)\cdot t^2}$$

    so:

    $$\frac{y-2}{y}\,dy = -\left(1 + \frac{1}{t} + t^{-2}\right)\,dt$$

    (NOTE: Now the DE is also exact).

    The solution is: $-y + 2\ln|y| = -\left(t + \ln|t| - \frac{1}{t} + C\right)$

14. $2xy^2 + 2y + (2x^2y + 2x)y' = 0$

    Exact. $x^2y^2 + 2xy = C$.

15. $x^3 \dfrac{dy}{dx} = 1 - 2x^2y$.

    Linear: $y' + \frac{2}{x}y = x^{-3}$, with integrating factor x^2:

$$y = \frac{\ln|x| + C}{x^2}$$

3 Misc.

1. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = t - 3 + \frac{C}{t^2}$$

   Construct y'. The idea will be to produce a linear DE. Therefore, we need to construct y' and compare it to y:

$$y' = 1 - 2Ct^{-3}$$

   Add this to some multiple (t's are allowed) of y to get of the arbitrary constant. In this case,

$$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

   or,

$$ty' + 2y = 3t - 6$$

2. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = 2\sin(3t) + C\mathrm{e}^{-2t}$$

$$y' = 6\cos(3t) - 2C\mathrm{e}^{-2t}$$

   so that: $y' + 2y = 4\sin(3t) + 6\cos(3t)$.

3. Construct an autonomous differential equation that has stable equilibria at $y(t) = 1$ and $y(t) = 3$, and one unstable equilibrium at $y(t) = 2$. (Hint: Draw the phase plot first).

   The formula would be something like:

$$y' = -\alpha(y - 1)(y - 2)(y - 3)$$

   with $\alpha > 0$.

4. Construct an autonomous differential equation so that all solutions tend towards $y(t) = 2$ as $t \to \infty$.

   (Compare to the quiz question). Using a phase plot, we see that any line with a negative slope and going through $(2, 0)$ will work for the DE. You could get more exotic with a shifted cubic if you wanted to, but we'll stay simple:

$$y' = m(y - 2), \qquad m < 0$$

5. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

   Write the initial value problem that describes the amount of salt in the tank at time t, and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \qquad Q(0) = Q_0$$

   The solution is:

$$Q = rM + (Q_0 - rM)\mathrm{e}^{-(\gamma/M)t}$$

6. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0? Does this make sense?

   Note that the differential equation for Q is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as $t \to \infty$ and see that $Q \to rM$. This does not necessarily depend on Q_0; if Q_0 starts at equilibrium, rM, then Q is constant.

   It does make sense. The incoming concentration of salt is r pounds per gallon, so we would expect the long term concentration to be the same, $rM/M = r$.

7. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

$$\frac{dQ}{dt} = 4 - \frac{3}{100+t}Q \qquad Q(0) = 50$$

   This goes from being autonomous to linear. In this case, use an integrating factor,

$$\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{3\int \frac{1}{100+t}\,dt} = \mathrm{e}^{3\ln|100+t|} = (100+t)^3 \qquad t > -100$$

   Continuing, we get:

$$Q(t) = -\frac{50,000,000}{(100+t)^3} + 100 + t$$

8. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t.

   The general model is: $mv' = mg - kv$. In this case, $m = 1$, $g = 9.8$ and $k = 1/2$. Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

   Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

   Solve it:

$$v(t) = 19.6\left(1 - \mathrm{e}^{-(1/2)t}\right)$$

9. (Continuing with the last problem): At $t = 10$ minutes, the force due to air resistance suddenly changes to $10v$. Model the velocity for $t \geq 10$ (set up and solve the IVP):

   The dynamics are now:

$$v' = 9.8 - 10v$$

   In order to make v continuous, the initial condition used here will be where the velocity left off after the last problem.

   If we make time re-start at zero (so that t is minutes after the previous 10), we would make $v(0) = 19.6\left(1 - \mathrm{e}^{-5}\right)$, which is approximately 19.467. The solution for t minutes after the original 10 minutes is:

$$v(t) = 0.98 + 18.487\mathrm{e}^{-10t}$$

   NOTE: If you did not restart time, the initial condition would be the same, except $v(10) \approx 19.467$, and the solution would be scaled:

$$v(t) = 0.98 + (18.487 \times \mathrm{e}^{100})\mathrm{e}^{-10t}$$

   valid for $t > 10$.

10. (Continuing with the falling object): In a direction field, draw a sketch of the solution. HINT: These are autonomous differential equations, so you should draw the phase plot first!

    To draw the direction field, first get the equilibrium solutions:

$$v = 2 \cdot 9.8 = 19.6 \qquad v = \frac{9.8}{10} = 0.98$$

    Your direction field should look something like Figure 1.

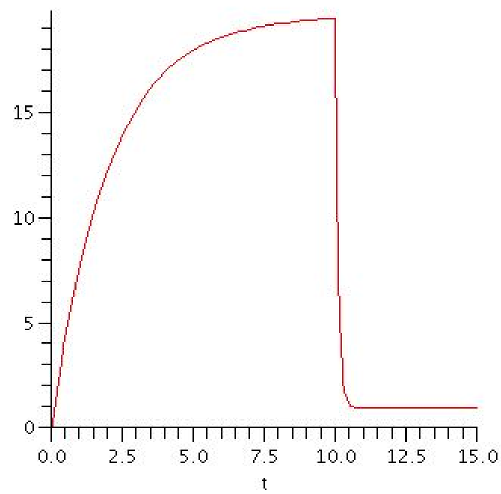

Figure 1: A graph of the velocity for the falling object problem. Note where the equilibria are: $v = 19.6$ for $t < 10$ and $v = 0.98$ for $t > 10$.

11. Suppose $y' = ky(1-y)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

    Your phase plot should be an upside down parabola intersecting with $y = 0$ and $y = 1$. We see that $y = 0$ is unstable and $y = 1$ is stable.

    (a) $y(t)$ is increasing if $0 < y_0 < 1$, decreasing otherwise.

    (b) For concavity, recall that, if $y' = f(y)$, then:

    $$\frac{d^2y}{dt^2} = \frac{df}{dy} \cdot \frac{dy}{dt} = \frac{df}{dy} \cdot f(y)$$

    Now, if $y < 0$, we see that $df/dy > 0$ and $f(y) < 0$, so y is concave down and decreasing.
    If $0 < y < 1/2$, then $df/dy > 0$ and $f(y) > 0$, so y is concave up and increasing.
    If $1/2 < y < 1$, then $df/dy < 0$ and $f(y) > 0$, so y is concave down and increasing.
    If $y > 1$, then $df/dy < 0$ and $f(y) < 0$, so y is concave up and decreasing.

    (c) The analytic solution (you'll need integration by parts):

    $$\int \frac{1}{y(1-y)}\,dy = \int k\,dt \quad \Rightarrow \quad \int \frac{1}{y} + \frac{1}{1-y}\,dy = kt + C \quad \Rightarrow \quad \ln|y| - \ln|1-y| = kt + C$$

    Now,

    $$\ln\left|\frac{y}{1-y}\right| = kt + C \quad \Rightarrow \quad \frac{y}{1-y} = Ae^{kt} \quad \Rightarrow \quad y(t) = \frac{Ae^{kt}}{1 + Ae^{kt}}$$

    In class, we simplified this to:

    $$y(t) = \frac{1}{(1/A)e^{-kt} + 1}$$

    which is easier to analyze.