# Selected Solutions: Reading HW, Chapter 1

2. A mathematical model is a DE that describes some physical process.
4. An equilibrium solution is a solution that never changes in time (a constant solution).
6. The trade off in modeling: (p. 15) Accuracy versus Simplicity.
8. The three important questions about ODEs: Existence of a solution, Uniqueness of a solution, Computability of a solution.

# Selected Solutions: Chapter 1

- Section 1.1, 1-5 odd: For the general situation, $y' = ay + b$, $y(0) = y_0$, we know the general solution from class:

  $$y(t) = Pe^{at} - \frac{b}{a}$$

  With $y(0) = y_0$, we see that $P = y_0 + \frac{b}{a}$. Therefore:

  Ex 1 $y' = 3 - 2y$, so $a = -2$ and $b = 3$. Since $e^{at} = e^{-2t}$, and this term goes to zero as $t \to \infty$, then $y(t)$ will always tend to $-b/a = 3/2$. The only exception is the equilibrium solution, $y = 3/2$ (If $y(0) = 3/2$, then $y(t) = 3/2$ for all $t$.

  The behavior does depend on the initial value: If $y_0 > 3/2$, the function $y(t)$ decreases to $3/2$ as $t \to \infty$. If $y_0 < 3/2$, the function increases to $3/2$, and if $y_0 = 3/2$, we stay at $3/2$.

  - The other problems have similar solutions.

- Section 1.1, #7: Using what we have just learned in 1-5 odd, $y' = -y + 3$ is one possibility.

- Section 1.1, #15-20 Matching. An easy way to check is to look for the equilibrium solutions- Where $y' = 0$. For example, ODE (d) would have equilibria at $0 = y(y+3)$, or $y = 0$ and $y = 3$. None of 15-20 have these on the direction field. On the other hand, something like ODE (c) has $y = 2$ as the only equilibrium solution, and all other solutions will tend away from it. Therefore, Exercise 16 is (c). Here are the others:

  $$15(j) \quad 16(c) \quad 17(g) \quad 18(b) \quad 19(h) \quad 20(e)$$

- Section 1.1, #22: Given

  $$V = \frac{4}{3}\Pi r^3 \qquad A = 4\pi r^2$$

  if $V' = c_1 A$, then we need to write $V$ in terms of $A$. Given the equations above,

  $$r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3} \quad \Rightarrow \quad A = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3} = kV^{2/3}$$

Therefore, $V' = c_1 A = c_2 V^{2/3}$.

- Section 1.2, 1(a,b): We'll go ahead and use the formula we got earlier:

$$y(t) = \left(y_0 + \frac{b}{a}\right) \mathrm{e}^{at} - \frac{b}{a}$$

Then the two solutions are:

$$y(t) = (y_0 - 5)\mathrm{e}^{-t} + 5 \qquad y(t) = \left(y_0 - \frac{5}{2}\right) \mathrm{e}^{-2t} + \frac{5}{2}$$

Both DEs have a single equilbrium towards which all solutions tend. Solutions to the second DE will tend towards its equilibrium much faster.

- Section 1.2, #3: If $a, b$ are both positive, we have seen that solutions to $y' = -ay + b$ will tend toward the equilibrium solution $y = b/a$.

- Section 1.2, #15: We have seen that the solution is $u(t) = (u_0 - T)\mathrm{e}^{-kt} + T$

  In part(b), consider the following statements and their translations:

  - The temperature difference at time $\tau$: $u(\tau) - T$.
  - The temperature difference at time 0: $u_0 - T$
  - Therefore,

  $$u(\tau) - T = \frac{1}{2}(u_0 - T)$$

  where $u(\tau) = (u_0 - T)\mathrm{e}^{-k\tau} + T$ (substitute in to get a relationship between $k$ and $\tau$.

- Section 1.3, #7: Practice with hyperbolic sine and cosine (Definitions below):

$$y_1 = \mathrm{e}^t \quad y_1' = \mathrm{e}^t \quad y_1'' = \mathrm{e}^t \quad \Rightarrow \quad y_1'' - y_1 = \mathrm{e}^t - \mathrm{e}^t = 0$$

With $y_2 = \cosh(t) = \frac{1}{2}(e^t + e^{-t})$, and $y_2' = \sinh(t)$ and $y_2'' = \cosh(t)$, we have:

$$y_2'' - y_2 = \cosh(t) - \cosh(t) = 0$$

- Section 1.3, #14: Recall from the Fundamental Theorem of Calculus that

$$\frac{d}{dt} \int_0^t g(s)\, ds = g(t)$$

Then use the product rule to differentiate the given $y$:

$$y = \mathrm{e}^{t^2} \int_0^t \mathrm{e}^{-s^2}\, ds + \mathrm{e}^{t^2}$$

$$y' = \left(2t\mathrm{e}^{t^2}\right)\int_0^t g(s)\,ds + \mathrm{e}^{t^2}\left(\mathrm{e}^{-t^2}\right) + 2t\mathrm{e}^{t^2}$$

Therefore, $y' - 2ty$ simplifies:

$$\left(2t\mathrm{e}^{t^2}\right)\int_0^t g(s)\,ds + \mathrm{e}^{t^2}\left(\mathrm{e}^{-t^2}\right) + 2t\mathrm{e}^{t^2} - 2t\left(\mathrm{e}^{t^2}\int_0^t \mathrm{e}^{-s^2}\,ds + \mathrm{e}^{t^2}\right) = \mathrm{e}^{t^2-t^2} = 1$$

- Section 1.3, #15-17: The basic idea here is that you want to be able to verify if a given model equation is a solution. In these cases, the model equation is $y = \mathrm{e}^{rt}$. In other exercises, it will be different.

  Generally speaking, if we substitute

  $$y = \mathrm{e}^{rt}, y' = r\mathrm{e}^{rt}, y'' = r^2\mathrm{e}^{rt}$$

  into $ay'' + by' + cy = 0$, we get:

  $$ar^2\mathrm{e}^{rt} + br\mathrm{e}^{rt} + c\mathrm{e}^{rt} = 0 \quad \Leftrightarrow \mathrm{e}^{rt}(ar^2 + br + c) = 0$$

  Then either $\mathrm{e}^{rt} = 0$ (no solution), or $ar^2 + br + c = 0$, which we solve by factoring or the quadratic formula. Specifically, for #15, we get:

  $$r + 2 = 0 \quad \Rightarrow \quad r = -2$$

  And for #17, we get:

  $$r^2 + r - 6 = 0 \quad \Rightarrow \quad (r+3)(r-2) = 0 \quad \Rightarrow \quad r = 2, -3$$

- Section 1.3, #19 and 20: Similar to #15, 17 except that the model equation is $y = t^r$, $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$.

  In #19, substituting into the DE we get

  $$r(r-1)t^r + 4rt^r + 2t^r = 0 \quad \Rightarrow \quad t^r\left(r(r-1) + 4r + 2\right) = 0$$

  Then, like before, solve the resulting quadratic for $r$ ($t^r \neq 0$, since $t > 0$).

- Section 1.3, #25:

  1. Given $u = \cos(x)\cosh(y)$ (subscript 1 removed for notation), then

     $$u_x = -\sin(x)\cosh(y) \qquad u_{xx} = -\cos(x)\cosh(y)$$

     and, using the relationship from earlier: $(\cosh(y))' = \sinh(y)$ and $(\sinh(y))' = \cosh(y)$,

     $$u_y = \cos(x)\sinh(y) \qquad u_{yy} = \cos(x)\cosh(y)$$

     Therefore, $u_{xx} + u_{yy} = 0$

2. Similarly, if $u_2 = \ln(x^2 + y^2)$, then (for the second derivative, use quotient rule and simplify):

$$u_x = \frac{2x}{x^2 + y^2} \qquad u_{xx} = \frac{-2(x^2 - y^2)}{(x^2 + y^2)^2}$$

And, $u_2$ is "symmetric" in $x, y$, so the derivatives in $y$ will look the same (just exchange the $x, y$):

$$u_y = \frac{2y}{x^2 + y^2} \qquad u_{yy} = \frac{-2(y^2 - x^2)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

and we see that $u_{xx} + u_{yy} = 0$.