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- In practice, it is much easier to measure the rate of change of a quantity rather than the quantity itself-
- A DE that describes a physical process is often called a **mathematical model**

We'll look at three models: Free fall, Owls and Mice, and Newton's Law of Cooling.

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$$mv' = mg - \gamma v \quad \Rightarrow \quad v' = g - \frac{\gamma}{m} v$$

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$$\frac{dP}{dt} = rP - k$$

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Let T be the environmental temp. Then:

$$\frac{du}{dt} = -k(u - T)$$

where k, T are positive (or zero).

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  where a, b are constants (unrestricted- could be negative or zero).
- All 3 models are of the same type, but represent very different physical phenomena- This is the power of using mathematical modeling.

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Definition: An equilibrium solution is a solution that does not change in time. Example: $y = -b/a$

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$$2 = P\mathrm{e}^0 - 2 \quad \Rightarrow \quad y(t) = 4\mathrm{e}^t - 2$$

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An **Initial Value Problem** (IVP) is a DE with an initial condition (IC).

Definitions:

- Ordinary vs Partial Differential Equations (ODE vs. PDE)
  Examples:
  $$u'' + u' + u = 3t \qquad\qquad u_{xx} + u_{yy} = 0$$
- Order of the DE: The order of the highest derivative.
  General first order ODE: $y' = f(t, y)$

More Definitions

- Linear ODE: Any ODE that can be written in the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = G(t)$$

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The first example is linear (and we gave a solution), the second is nonlinear (and we cannot find an elementary solution).

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- A second question is one of *uniqueness*: If the ODE has a solution, does it have more than one?
- Third is a practical question: If the ODE has a solution, can we compute it?

We'll answer these questions this semester....

Consider this:

NOTE

The study of “nonlinear ODEs” is like the study of “non-elephant animals” in the sense that often, the real world is nonlinear. Often, analytic solutions will not be possible.

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Example: $y' = t - y^2$
In drawing a picture, we might consider curves of constant slope. For example, with zero slope:

$$0 = t - y^2 \quad \Rightarrow \quad y^2 = t$$

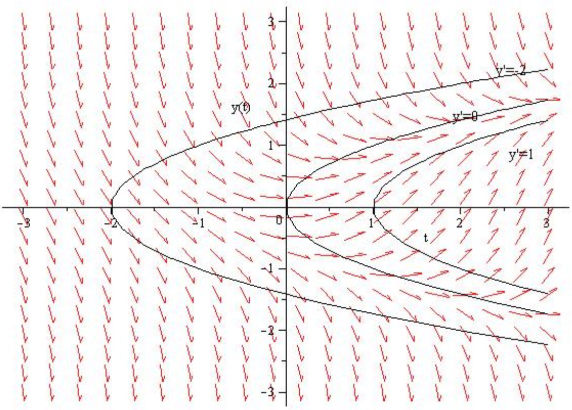
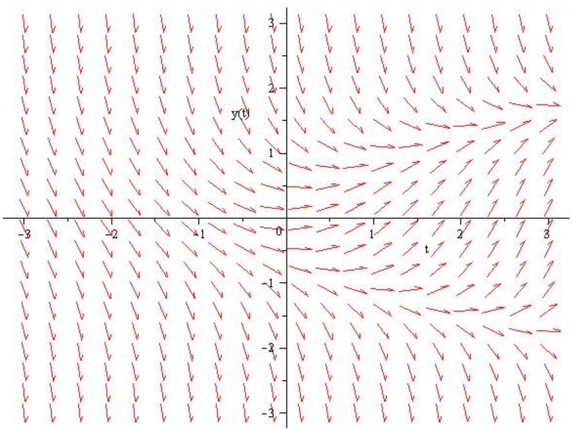
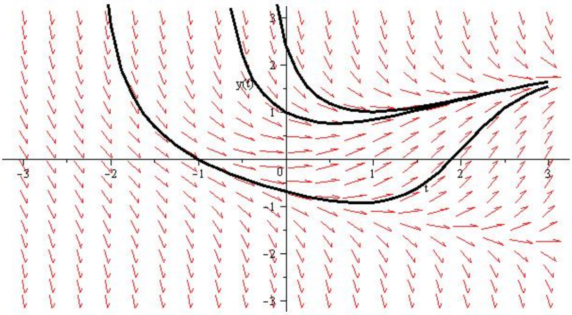


Figure: Direction Field with Isoclines: $y' = -2, y' = 0, y' = 1$


y(t)
t
3
2
1
0
-1
-2
-3

Graphically determine the solutions (integral curves) to $y' = t - y^2$ for the following initial conditions:

$$y(0) = 1 \qquad y(-1) = 0 \qquad y(1) = 1$$



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Give the solution and the direction field for the falling body, if the mass is 10 kg and $\gamma = 2$ kg/s (and $m = 9.8$ m/s^2)

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The other solutions are $(a = -1/5,\ b = 9.8)$

$$v(t) = P\mathrm{e}^{-t/5} + 49$$

(Draw the direction field and show the solutions)

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so that $k = 2$ or $k = -3$

Homework Hint: #14, Section 1.3

Differentiate the following with respect to t:

$$f(t)\int_0^t G(s)\,ds$$

SOLUTION:

$$f'(t)\int_0^t G(s)\,ds + f(t)G(t)$$

Homework Note: # 25, 1.3

If you know $\cosh(x)$, go ahead. If not, use only u_2 instead of u_1 and u_2.

Recall the following techniques:

- Partial Fractions
- Integration by parts