Summary of Chapter 3

We can think of the chapter as being split into two: General theory, and Computation. First, the general theory.

General Theory, Chapter 3

The goal of the theory was to establish the structure of solutions to the second order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

We saw that two functions form a fundamental set of solutions to the homogeneous DE if the Wronskian is not zero (at the initial value of time).

1. Vocabulary: Operator, Linear Operator, general solution, fundamental set of solutions.

2. Theorems:

   - The Existence and Uniqueness Theorem for $y''+p(t)y'+q(t)y = g(t)$: If there is an open interval I on which p, q and g exists, and if I contains the initial time t_0, then there exists a unique solution to the IVP, valid on I.
   - Principle of Superposition: If L is a linear operator, and y_1, y_2 are two functions so that $L(y_1) = 0$ and $L(y_2) = 0$, then so does any function of the form $c_1y_1 + c_2y_2$.
   - Abel's Theorem.
     If y_1, y_2 are solutions to $y''+p(t)y'+q(t)y = 0$, then the Wronskian is either always zero or never zero on the interval for which the solutions are valid.
     That is because the Wronskian may be computed as:
     $$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$
   - The Fundamental Set of Solutions: $y'' + p(t)y' + q(t)y = 0$
     We can guarantee that we can always find a fundamental set of solutions. We did that by appealing to the Existence and Uniqueness Theorem for the following two initial value problems:
     - y_1 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 1, y'(t_0) = 0$
     - y_2 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 0, y'(t_0) = 1$

3. The Structure of Solutions to $y'' + p(t)y' + q(y)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$

   Given that y_h solves the homogeneous equation, and y_p solves the forced equation, then the general solution to the forced equation is
   $$y_h + y_p$$
   Or, we can be much more specific:

   Given a fundamental set of solutions to the homogeneous equation, y_1, y_2, then there is a solution to the initial value problem, written as:
   $$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$
   where $y_p(t)$ solves the non-homogeneous equation.

   In fact, if we have:
   $$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t)$$
   we can solve by splitting the problem up into smaller problems:

- y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
- y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
- y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
  and so on..
- y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

and the full solution is:

$$y(t) = C_1 y_1 + C_2 y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

Computation of Solutions, Chapter 3

From the theory, we know that every initial value problem:

$$ay'' + by' + cy = g(t) \qquad y(t_0) = y_0 \quad y'(t_0) = v_0$$

has a solution that can be expressed as:

$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where y_1, y_2 form a fundamental set of solutions to the homogeneous equation, and $y_p(t)$ is the (particular) solution to the nonhomogeneous equation.
We first consider the homogeneous ODE:

Solving $ay'' + by' + cy = 0$

Form the associated characteristic equation (built by using $y = \mathrm{e}^{rt}$ as the ansatz):

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that the solutions depend on the discriminant, $b^2 - 4ac$ in the following way (y_h refers to the solution of the homogeneous equation):

- $b^2 - 4ac > 0 \Rightarrow$ 2 distinct real roots r_1, r_2. $y_h(t) = c_1 \mathrm{e}^{r_1 t} + c_2 \mathrm{e}^{r_2 t}$
- $b^2 - 4ac = 0 \Rightarrow$ one real root $r = -b/2a$: $y_h(t) = \mathrm{e}^{-(b/2a)t} (C_1 + C_2 t)$
- $b^2 - 4ac < 0 \Rightarrow$ 2 complex solutions, $r = \lambda \pm i\mu$: $y_h(t) = \mathrm{e}^{\lambda t} (C_1 \cos(\mu t) + C_2 \sin(\mu t))$

Solving $y'' + p(t)y' + q(t)y = 0$

Given $y_1(t)$, we can solve for a second linearly independent solution to the homogeneous equation, y_2, by one of two methods:

- By use of the Wronskian: There are two ways to compute this,
  - $W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$ (This is from Abel's Theorem)
  - $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

  Therefore, these are equal, and y_2 is the unknown: $y_1 y_2' - y_2 y_1' = C\mathrm{e}^{-\int p(t)\,dt}$
- Reduction of order, where $y_2 = v(t) y_1(t)$.

Finding the particular solution.

Our two methods were: Method of Undetermined Coefficients and Variation of Parameters.

- Method of Undetermined Coefficients

  This method is motivated by the observation that, a linear operator of the form $L(y) = ay'' + by' + cy$, acting on certain classes of functions, returns the same class. In summary, the table from the text:

if $g_i(t)$ is:	The ansatz y_{p_i} is:
$P_n(t)$	$t^s(a_0 + a_1 t + \ldots a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_0 + a_1 t + \ldots + a_n t^n)$
$P_n(t)\mathrm{e}^{\alpha t}\sin(\mu t)$ or $\cos(\mu t)$	$t^s \mathrm{e}^{\alpha t}\left((a_0 + a_1 t + \ldots + a_n t^n)\sin(\mu t)\right.$ $\left.+ (b_0 + b_1 t + \ldots + b_n t^n)\cos(\mu t)\right)$

  The t^s term comes from an analysis of the homogeneous part of the solution. That is, multiply by t or t^2 so that no term of the ansatz is included as a term of the homogeneous solution.

- Variation of Parameters: Given $y'' + p(t)y' + q(t)y = g(t)$, with y_1, y_2 solutions to the homogeneous equation, we write the ansatz for the particular solution as:

$$y_p = u_1 y_1 + u_2 y_2$$

  From our analysis, we saw that u_1, u_2 were required to solve:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

  From which we get the formulas for u_1' and u_2':

$$u_1' = \frac{-y_2 g}{W(y_1, y_2)} \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$