Solutions to Extra Practice Problems
Linear Operators and Cramer's Rule

1. Let $R(f)$ be the operator defined by: $R(f) = f''(t) + 3t^2 f(t)$. Find $R(f)$ for each function below:

   (a) $f(t) = t^2$: Since $f'(t) = 2t$ and $f''(t) = 2$, we have:
   $$R(t^2) = 2 + 3t^2 \cdot t^2 = 2 + 3t^4$$

   (b) $f(t) = \sin(3t)$: Since $f'(t) = 3\cos(3t)$ and $f''(t) = -9\sin(3t)$, we substitute:
   $$R(\sin(3t)) = -9\sin(3t) + 3t^2 \sin(3t) = (3t^2 - 9)\sin(3t)$$

   (c) $f(t) = 2t - 5$
   $$R(2t-5) = 0 + 3t^2(t-5) = 3t^3 - 15t^2$$

2. Let R be the operator defined in the previous problem. Show that R is a linear operator.

   - $R(f+g) = (f+g)'' + 3t^2(f+g) = f'' + g'' + 3t^2 f + 3t^2 g = f'' + 3t^2 f + g'' + 3t^2 g = R(f) + R(g)$
   - $R(cf) = (cf)'' + 3t^2(cf) = cf'' + 3t^2 cf = c(f'' + 3t^2 f) = cR(f)$

3. Let $F(y) = y'' + y - 5$. Explain why F is not linear.

   Best way is to show it- This F does not satisfy either part of the definition. For example,

   $F(x+y) = (x+y)'' + (x+y) - 5 = x'' + x + y'' + y - 5 \neq (x'' + x - 5) + (y'' + y - 5) = F(x) + F(y)$

4. Find the operator associated with the given differential equation, and classify it as linear or not linear:

   (a) $y' = ty^2 + \cos(t)$
   $$L(y) = y' - ty^2$$
   Not linear

   (b) $y'' = 4y' + 3y + \sin(t)$
   $$L(y) = y'' - 4y' - 3y$$
   Linear

   (c) $y' = e^t y + 5$
   $$L(y) = y' - e^t y$$
   Linear (in y)

   (d) $y'' = -\cos(y) + \cos(t)$
   $$L(y) = y'' + \cos(y)$$
   Not linear

5. Use Cramer's Rule to solve the following systems:

   (a) $\begin{aligned} C_1 + C_2 &= 2 \\ -2C_1 - 3C_2 &= 3 \end{aligned}$ SOLUTION:

   $$C_1 = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{-9}{-1} = 9 \qquad C_2 = \frac{\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{7}{-1} = -7$$

(b) $$\begin{aligned} C_1 + C_2 &= y_0 \\ r_1 C_1 + r_2 C_2 &= v_0 \end{aligned}$$

$$C_1 = \frac{r_2 y_0 - v_0}{r_2 - r_1} \qquad C_2 = \frac{v_0 - r_1 y_0}{r_2 - r_1}$$

(c) $$\begin{aligned} C_1 + C_2 &= 2 \\ 3C_1 + C2 &= 1 \end{aligned}$$

$$C_1 = -\frac{1}{2} \qquad C_2 = \frac{5}{2}$$

(d) $$\begin{aligned} 2C_1 - 5C_2 &= 3 \\ 6C1 - 15C_2 &= 10 \end{aligned}$$ The denominator to Cramer's Rule is zero- No solution.

(e) $$\begin{aligned} 2x - 3y &= 1 \\ 3x - 2y &= 1 \end{aligned}$$

$$x = 1, y = 1$$

6. Suppose L is a linear operator. Let y_1, y_2 each solve the equation $L(y) = 0$ (so that $L(y_1) = 0$ and $L(y_2) = 0$). Show that anything of the form $c_1 y_1 + c_2 y_2$ will also solve $L(y) = 0$.

   SOLUTION:

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

   Go back to Section 2.4, page 76 and look at Exercises 23-26. This section generalizes those results.