Homework Set 1 Solutions

1. Exercise 22, p 363 (Section 7.1)

   Construct the DE's by looking at "Rate In - Rate Out", and make sure your units are matching up. Define $Q_1(t)$ and $Q_2(t)$ to be the ounces of salt in Tanks 1 and 2 (respectively). Before we start, we take note of the rates in and out of each tank- The total amounts of water in tanks 1 and 2 (30 gallons and 20 gallons, respectively) does not change (important for the rates in and out).

   Filling in the quantities from the figure on p. 363, we have:

$$\frac{dQ_1}{dt} = \frac{1.5 \text{ gal}}{\text{min}} \cdot \frac{1 \text{ oz}}{\text{gal}} + \frac{1.5 \text{ gal}}{\text{min}} \cdot \frac{Q_2 \text{ oz}}{20 \text{ gal}} - \frac{3 \text{ gal}}{\text{min}} \cdot \frac{Q_1 \text{ oz}}{30 \text{ gal}}$$

$$\frac{dQ_2}{dt} = \frac{1 \text{ gal}}{\text{min}} \cdot \frac{3 \text{ oz}}{\text{gal}} + \frac{3 \text{ gal}}{\text{min}} \cdot \frac{Q_1 \text{ oz}}{30 \text{ gal}} - \frac{4 \text{ gal}}{\text{min}} \cdot \frac{Q_2 \text{ oz}}{20 \text{ gal}}$$

   Simplifying, and putting them in order:

$$\frac{dQ_1}{dt} = -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2}$$

$$\frac{dQ_2}{dt} = \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3$$

   The system is currently non-homogeneous because of the constants 3/2 and 3.

   The equilibria are found by solving for where the derivatives are zero. We simplify to make Cramer's Rule easier to apply:

$$\begin{array}{rl} -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2} & = 0 \\ \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} -4Q_1 + 3Q_2 & = -60 \\ Q_1 - 2Q_2 & = -30 \end{array}$$

   Therefore, using the determinants, the equilibrium solution is:

$$Q_1 = \frac{210}{5} = 42 \qquad Q_2 = \frac{180}{5}$$

   The last part of the question shows that a change of variables results in a homogeneous differential equation. That is, if $x_1 = Q_1 - 42$ and $x_2 = Q_2 - 36$, then substitution into the system of DE's:

$$x_1' = Q_1' \qquad x_2' = Q_2'$$

   and

$$-\frac{1}{10}(x_1 + 42) + \frac{3}{40}(x_2 + 36) + \frac{3}{2} = -\frac{1}{10}x_1 + \frac{3}{40}x_2$$

   Similarly, substitution into the second equation and substituting:

$$\frac{1}{10}(x_1 + 42) - \frac{1}{5}(x_2 + 36) + 3 = \frac{1}{10}x_1 - \frac{1}{5}x_2$$

   Notice that our change of coordinates simply shifted the coordinate system so that the equilibrium is now at the origin. To get the initial conditions, make the last substitution:

$$\begin{array}{rl} x_1' & = -\frac{1}{10}x_1 + \frac{3}{40}x_2 \\ x_2' & = \frac{1}{10}x_1 - \frac{1}{5}x_2 \end{array} \qquad x_1(0) = -17, \quad x_2(0) = -21$$

2. Text questions from Section 7.2

1.

$$2A+B=\begin{bmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{bmatrix} \qquad AB=\begin{bmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{bmatrix}$$

1(b) and (d) are done in a similar fashion.

4. Done in class.

22. Verify the solution given:

$$\mathbf{x}'(t)=\begin{bmatrix} 4 \\ 2 \end{bmatrix} 2\mathrm{e}^{2t}=\begin{bmatrix} 8 \\ 4 \end{bmatrix}\mathrm{e}^{2t}$$

And

$$A\mathbf{x}=\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}\mathrm{e}^{2t}=\begin{bmatrix} (3)(4)-(2)(2) \\ (2)(4)-(2)(2) \end{bmatrix}\mathrm{e}^{2t}=\begin{bmatrix} 8 \\ 4 \end{bmatrix}\mathrm{e}^{2t}$$

23. We can re-write these equations. The system of DEs is:

$$\begin{aligned} x_1' &= 2x_1-x_2+\mathrm{e}^t \\ x_2' &= 3x_1-2x_2-\mathrm{e}^t \end{aligned} \qquad \text{where} \qquad \begin{aligned} x_1 &= \mathrm{e}^t+2t\mathrm{e}^t=\mathrm{e}^t(1+2t) \\ x_2 &= 2t\mathrm{e}^t \end{aligned}$$

Make the substitutions to see that these functions satisfy the DEs. For example,

$$2x_1-x_2+\mathrm{e}^t=2\mathrm{e}^t+4t\mathrm{e}^t-2t\mathrm{e}^t+\mathrm{e}^t=\mathrm{e}^t(3+2t)$$

while using the product rule gives:

$$x_1'=\mathrm{e}^t(1+2t)+2\mathrm{e}^t=\mathrm{e}^t(3+2t)$$

For each of the problems below, define

$$A=\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{b}=\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad B=\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and I is the identity matrix:

3. Compute $(B-4I)\boldsymbol{b}$

$$\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}-\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\mathbf{b}=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

4. Compute $\det(B-4I)=0$

5. Are AB and BA the same? No:

$$AB=\begin{bmatrix} 5 & 7 \\ -2 & 2 \end{bmatrix} \qquad BA=\begin{bmatrix} 2 & 7 \\ -2 & 5 \end{bmatrix}$$

6. Compute $A^{-1}\boldsymbol{b}=\dfrac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

7. Verify: $A\boldsymbol{b}-3\boldsymbol{b}=(A-3I)\boldsymbol{b}$

8. Compute $B^T B$, $\mathrm{Tr}(A)$, and $\mathrm{Tr}(B)$

$$B^T B = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \quad \mathrm{Tr}(A) = 2 \quad \mathrm{Tr}(B) = 6$$

9. Short Answer:

   (a) Is every second order linear homogeneous differential equation (with constant coefficients) equivalent to a system of first order equations? **Yes.**

   (b) Can every 2×2 system of DEs be converted into an equivalent second order system? (Hint: To do our technique, what must be true?)

   No. In particular, one of the variables must be solvable in terms of the other. Example where this does not work:

$$\begin{aligned} x_1' &= 3x_1 \\ x_2' &= -x_2 \end{aligned}$$

10. Give the solution to each system. If it has an infinite number of solutions, give your answer in vector form:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned} \quad \Rightarrow \quad x = 1, y = -1$$

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 3 \end{aligned} \qquad \text{No Solution.}$$

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 2 \end{aligned} \quad \Rightarrow \quad y = -\frac{3}{2}x + \frac{1}{2} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

11. Write each of the previous systems in matrix-vector form. Verify that the determinant of the first matrix is not zero, but is zero for the second and third.

    Straightforward.

12. Write each system of differential equations in matrix-vector form or write the system from the matrix-vector form:

$$\begin{aligned} x_1' &= 3x_1 - x_2 \\ x_2' &= 9x_1 - 3x_2 \end{aligned} \qquad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

    Straightforward.

13. Find the equilibrium solutions to the previous autonomous linear differential equations.

    The determinant of the first is zero, which means we have an infinite number of solutions, $3x_1 - x_2 = 0$, or $x_2 = 3x_1$. In vector form, the solutions are:

$$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

    In the second DE, the determinant of the coefficients is not zero, so the only equilibrium is the origin.

14. If $\mathbf{x}$ is as defined below, compute $\mathbf{x}'(t)$, and $\int_0^1 \mathbf{x}(t)\,dt$:

$$\mathbf{x}(t) = \begin{bmatrix} t^2 - 3 \\ 3\mathrm{e}^t - 2\mathrm{e}^{3t} \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 2t \\ 3\mathrm{e}^t - 6\mathrm{e}^{3t} \end{bmatrix} \qquad \int_0^1 \mathbf{x}(t)\,dt = \frac{1}{3} = \begin{bmatrix} -8 \\ 9\mathrm{e} - 2\mathrm{e}^3 - 7 \end{bmatrix}$$

15. If $\mathbf{y}(t) = A(t)\mathbf{c}$ is as defined below, compute $\mathbf{y}'(t)$, and $\int_0^1 \mathbf{y}(t)\,dt$. Are these the same as $A'(t)\mathbf{c}$ and $\int A(t)\,dt\,\mathbf{c}$?

$$\mathbf{y}(t) = \begin{bmatrix} t & 2t \\ 1 & \sin(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

SOLUTION:

$$\begin{array}{llll} y_1(t) & = c_0 t + 2c_1 t \quad \Rightarrow & y_1'(t) = c_0 + 2c_1 \Rightarrow & \int_0^1 y_1(t)\,dt = \frac{1}{2}c_0 + c_1 \\ y_2(t) & = c_0 + c_1 \sin(t) \quad \Rightarrow & y_2'(t) = c_1 \cos(t) \Rightarrow & \int_0^1 y_2(t)\,dt = c_0 + c_1 - c_1 \cos(1) \end{array}$$

And

$$A'(t)\mathbf{c} = \begin{bmatrix} 1 & 2 \\ 0 & \cos(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_0 + 2c_1 \\ c_1 \cos(t) \end{bmatrix}$$

Finally

$$\int_0^1 A(t)\,dt\mathbf{c} = \begin{bmatrix} 1/2 & 2 \\ 1 & -\cos(1) + 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} (1/2)c_0 + 2c_1 \\ c_0 + c_1(-\cos(1) + 1) \end{bmatrix}$$

Yes, they are the same.

16. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

   (a) $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

   From the first equation, we get $x_2 = x_1' + 2x_1$. Substitute this into the second equation to get:

$$x_1'' + 2x_1' = x_1 - 2(x_1' + 2x_1) \quad \Rightarrow \quad x_1'' + 4x_1' + 3x_1 = 0$$

   We get a characteristic equation: $r^2 + 4r + 3 = 0$, and:

$$x_1 = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t} \qquad x_2 = x_1' + 2x_1 = C_1\mathrm{e}^{-t} - C_2\mathrm{e}^{-3t}$$

   Recall that we can verify our solution (not necessary, but good for practice):

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -C_1\mathrm{e}^{-t} - 3C_2\mathrm{e}^{-3t} \\ -C_1\mathrm{e}^{-t} + 3C_2\mathrm{e}^{-3t} \end{bmatrix}$$

   And, using the matrix, we get the same vector:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t} \\ C_1\mathrm{e}^{-t} - C_2\mathrm{e}^{-3t} \end{bmatrix}$$

(b) $\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$

Following our usual procedure, use one equation to solve for one of the variables. Using the first equation, we have: $x_2 = \frac{1}{2}x_1'$. Substitute this into the second to get a DE in terms of x_1 alone:

$$x_2' = -2x_1 \quad \Rightarrow \quad \frac{1}{2}x_1' = -2x_2 \quad \Rightarrow \quad x_1'' + 4x_1 = 0$$

The characteristic equation gives $r = \pm 2i$, so

$$x_1 = C_1 \cos(2t) + C_2 \sin(2t)$$

and $x_2 = \frac{1}{2}x_1' = -C_1 \sin(2t) + C_2 \cos(2t)$.

17. Convert the following second order differential equations into a system of autonomous, first order equations. Using methods from Chapter 3, give the solution to the system. An example follows before the exercises:

$$y'' + 3y' + 2y = 0$$

SOLUTION: We'll get the homogenous solution first. The roots to the characteristic equation are $-1, -2$. The general solution is:

$$y = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$$

To get an equivalent system, let $x_1 = y$ and $x_2 = y'$. Then

$$x_1' = y' = x_2 \qquad x_2' = y'' = -2y - 3y' = -2x_1 - 3x_2$$

so the system is (in matrix-vector form):

$$\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

Since $x_1 = y$, then $x_1 = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-2t}$. Since $x_2 = y'$, then $x_2 = -C_1\mathrm{e}^{-t} - 2C_2\mathrm{e}^{-2t}$. In vector form, this means our solution is:

$$\mathbf{x} = C_1\mathrm{e}^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Here we go:

(a) $y'' + 4y' + 3y = 0$ The solution is: $y = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t}$. Using our usual substitution $x_1 = y$, $x_2 = y'$, we have:

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -3y - 4y' = -3x_1 - 4x_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And we can write the solution as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{-3t} \\ -C_1\mathrm{e}^{-t} - 3C_2\mathrm{e}^{-3t} \end{bmatrix} = C_1\mathrm{e}^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\mathrm{e}^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

(b) $y'' + 5y' = 0$ The solution is: $y = C_1 + C_2\mathrm{e}^{-5t}$. The system, with the usual substitutions, is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solution can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1 + C_2\mathrm{e}^{-5t} \\ -5C_2\mathrm{e}^{-5t} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\mathrm{e}^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

(c) $y'' + 4y = 0$ We've seen one like this already in 14(b): The solution is $y = C_1 \cos(2t) + C_2 \sin(2t)$. Using the substitutions $x_1 = y, x_2 = y'$, we have:

$$x_1' = x_2 \quad x_2' = -4y = -4x_1 \quad \Rightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and we can write the solution as a vector (but not as neatly as before:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

(d) $y'' - 2y' + y = 0$ The characteristic equation gives $r = 1, 1$, so

$$y(t) = C_1 \mathrm{e}^t + C_2 t \mathrm{e}^t$$

Using the usual substitutions, the matrix-vector form of the first order system is:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Writing the solution as a system, we have:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1 \mathrm{e}^t + C_2 t \mathrm{e}^t \\ C_1 \mathrm{e}^t + C_2 \mathrm{e}^t + C_2 t \mathrm{e}^t \end{bmatrix}$$

For future reference, notice that this can be written as:

$$C_1 \mathrm{e}^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^t \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$