Notes (Ch 9): Poincare Classification

To solve $\mathbf{x} = A\mathbf{x}$, we compute the eigenvalues then the eigenvectors, and build the solution. We can also summarize the geometric behavior of the solutions by looking at a plot- However, there is an easier way to classify the origin (as an equilibrium),

To find the eigenvalues, we compute the characteristic equation:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}$$

which depends on the discriminant Δ:

- $\Delta > 0$: Real λ_1, λ_2.
- $\Delta < 0$: Complex $\lambda = a + ib$
- $\Delta = 0$: One eigenvalue.

The type of solution depends on Δ, and in particular, where $\Delta = 0$:

$$\Delta = 0 \quad \Rightarrow \quad 0 = (\text{Tr}(A))^2 - 4\det(A)$$

This is a parabola in the $(\text{Tr}(A), \det(A))$ coordinate system.

Example:

Given the system where $\mathbf{x}' = A\mathbf{x}$ for each matrix A below, classify the origin using the Poincaré Diagram:

1. $\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$\text{Tr}(A) = -6 \qquad \text{Det}(A) = 9 \qquad \Delta = 36 - 4 \cdot 9 = 0$$

   Therefore, we have a "degenerate sink". That is, we have a sink, and we have a degenerate matrix.

2. $\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$\text{Tr}(A) = 0 \qquad \text{Det}(A) = 9 \qquad \Delta = 0^2 - 4 \cdot 9 = -36$$

   We have a saddle (in fact, $\lambda = \pm 3$).

Example:

Given the system $\mathbf{x}' = A\mathbf{x}$ where the matrix A depends on α, describe how the equilibrium solution changes depending on α (use the Poincaré Diagram):

1. $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$

   SOLUTION: The trace is 0, so that puts us on the "det(A)" axis. The determinant is $-4 + 5\alpha$. If this is positive, we have a center:

   $$-4 + 5\alpha > 0 \quad \Rightarrow \quad \alpha > \frac{4}{5}$$

   If this is negative, we have a saddle:

   $$\alpha < \frac{4}{5}$$

   If $\alpha = \frac{4}{5}$, we have "uniform motion". That is, $x_1(t)$ and $x_2(t)$ will be linear in t (see if you can find the general solution!).

2. $\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$

SOLUTION: The trace is 2α and the determinant is $\alpha^2 + 1$. The discriminant is:

$$4\alpha^2 - 4(\alpha^2 + 1) = 4\alpha^2 - 4\alpha^2 - 4 = -4$$

Therefore, we always have a center (periodic solutions).

Linearizing a Nonlinear System

The following notes are elements from Sections 9.2 and 9.3.

- Suppose we have an autonomous system of equations:

  $$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

  Then (as before) we define a point (a, b) to be an **equilibrium point** for the system if $f(a, b) = 0$ AND $g(a, b) = 0$ (that is, you must solve the system of equations, not one at a time).

- **Example:** Find the equilibria to:

  $$\begin{aligned} x' &= -(x - y)(1 - x - y) \\ y' &= x(2 + y) \end{aligned}$$

  SOLUTION: From the second equation, either $x = 0$ or $y = -2$. Take each case separately.

- If $x = 0$, then the first equation becomes $y(1-y)$, so $y = 0$ or $y = 1$. So far, we have two equilibria:

$$(0,0) \qquad (0,1)$$

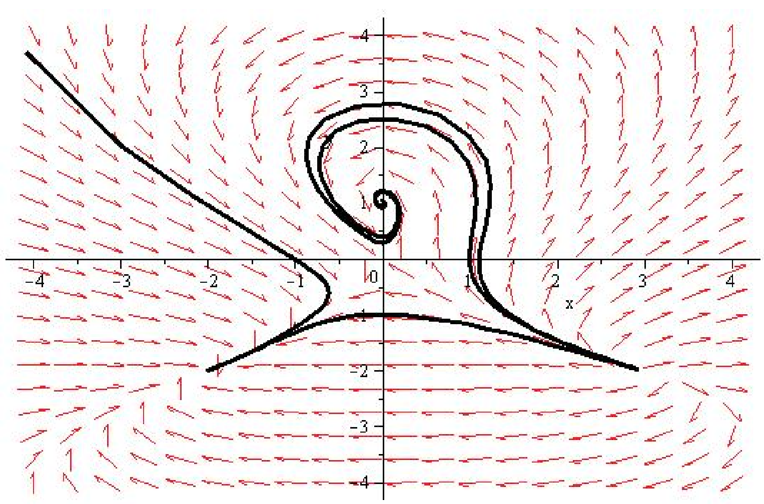
- Next, if $y = -2$ in the second equation, then the first equation becomes

$$-(x+2)(1-x+2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } x = 3$$

We now have two more equilibria:

$$(-2,-2) \qquad (3,-2)$$

- **Key Idea:** The "interesting" behavior of a dynamical system is organized around its equilibrium solutions.
- To see what this means, here is the graph of the direction field for the example nonlinear system:



- In order to understand this picture, we will need to linearize the differential equation about its equilibrium.
- Let $x = a, y = b$ be an equilibrium solution to $x' = f(x,y)$ and $y' = g(x,y)$. Then the linearization about (a,b) is the system:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where $u = x - a$ and $y = v - b$. In our analysis, we really only care about this matrix- You may have used it before, it is called the Jacobian matrix.

- Continuing with our previous example, we compute the Jacobian matrix, then we will insert the equilibria one at a time and perform our local analysis. We then try to put together a global picture of what's happening.

  Recall that the system is:

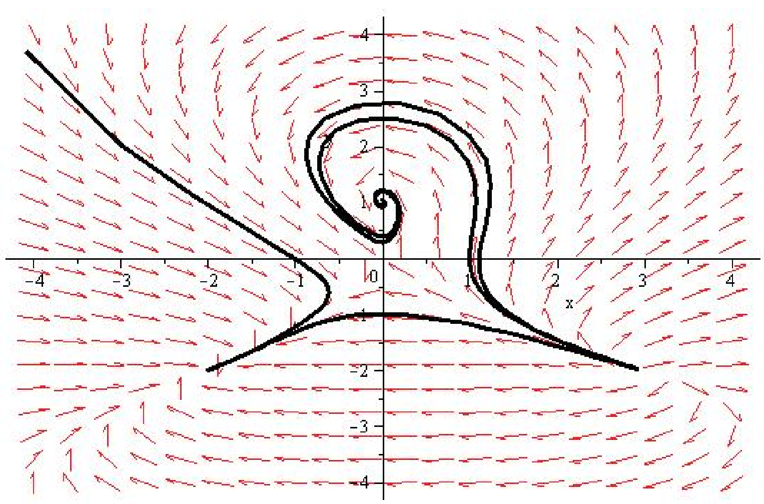
$$\begin{aligned} x' &= -(x-y)(1-x-y) &&= -x + x^2 + y - y^2 \\ y' &= x(2+y) &&= 2x + xy \end{aligned}$$

  The Jacobian matrix for our example is:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1+2x & 1-2y \\ 2+y & x \end{bmatrix}$$

Equilibrium	System	Tr(A)	det(A)	Δ	Poincare
$(0,0)$	$\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$	-1	-2		Saddle
$(0,1)$	$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$	-1	3	-11	Spiral Sink
$(-2,-2)$	$\begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$	-7	10	9	Sink
$(3,-2)$	$\begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$	8	15	4	Source

  Here's the picture again:



System	Tr(A)	det(A)	Δ	Poincare Fill in	λ	V
$\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$	0	9	-36		$3i$	$\begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$	4	4	0		$2, 2$	$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$	-1	$5/4$	-4		$-\frac{1}{2}+i$	$\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$	$-5/4$	$1/4$	$9/16$		$-1, -1/4$	$\begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

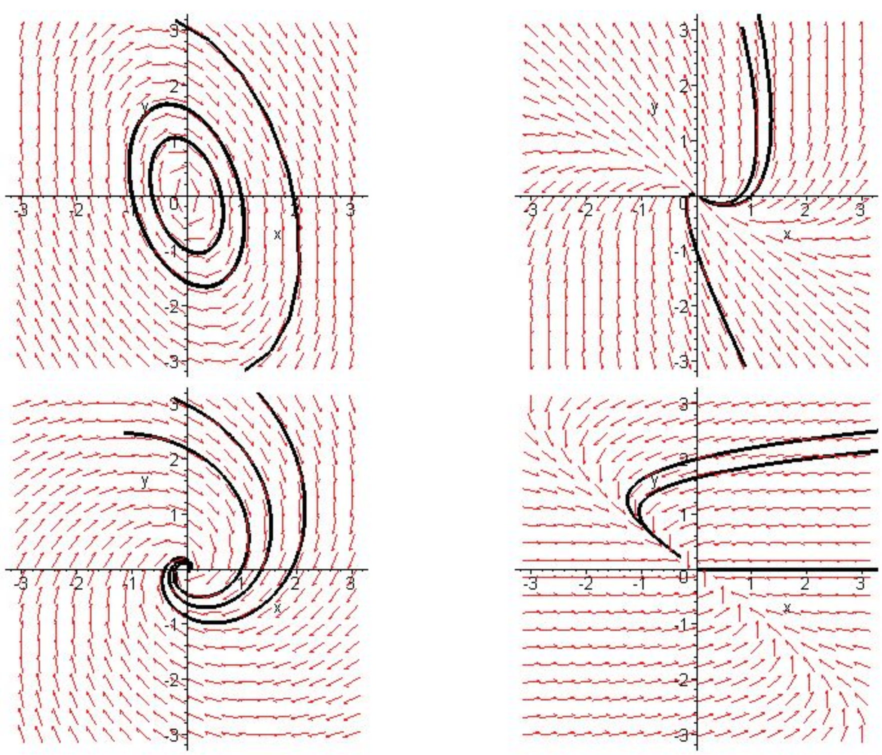


Figure 1: Phase planes. From top to bottom, Center, Degenerate Source, Spiral Sink, Sink.

Homework: Elements of Chapter 9, Day 1

1. Fill in the following and under "Poincaré" classify the origin. Then, given the eigenvalues/eigenvectors, also write down the general solution to $\mathbf{x}' = A\mathbf{x}$. In the case that there is only one eigenvector, the second column of V shows the generalized eigenvector $\mathbf{w}$.

System	Tr(A)	det(A)	Δ	Poincare Fill in	λ	V
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$					$1+2i$	$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$					$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$					$2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$					$0, 0$	$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}$

2. Explain how the classification of the origin changes by changing the α in the system:

   (a) $\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$

   (b) $\mathbf{x}' = \begin{bmatrix} 2 & \alpha \\ 1 & -1 \end{bmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix} \mathbf{x}$

   Hint: Use a number line to keep track of where the trace, determinant and discriminant change sign (reminiscent of sign charts in Calc I). This is Exercise 19, pg. 410 if you want to see the textbook solution- It is easier to organize the different possibilities using the Poincaré Diagram, however.

3. For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to t, as usual).

   (a) $x' = x - xy$, $y' = y + 2xy$

   (b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$