Solutions: Nonlinear Analysis Exercises

1. For each system below, (i) Find all equilibrium solutions, (ii) Linearize the DE about each, and (iii) Use the Poincaré Diagram to analyze the stability. Finally, go online to the phase plane plotter and see if your analysis is correct.

   (a) $\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$

   i. The equilibria are $(1, 1)$ and $(-1, 1)$.

   ii. The Jacobian matrix of first partial derivatives is: $\begin{bmatrix} 0 & -1 \\ 2x & 2y \end{bmatrix}$

   Applying this to $(1, 1)$: $\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$ (Saddle)

   Applying this to $(-1, 1)$: $\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ (Sink)

   iii. See figure below

   (b) $\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$

   i. The equilibria occur where x can be any multiple π, and y any odd multiple of $\pi/2$.

   ii. The Jacobian matrix of first partial derivatives is: $\begin{bmatrix} 0 & -\sin(y) \\ \cos(x) & 0 \end{bmatrix}$

   After some analysis, there are two types of equilibria. Type 1 has one of these matrices, which both correspond to *centers*:

   $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   The second type is one of the following, which are both *saddles*.

   $$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   The $\cos(x)$ term is $+1$ when $x = 2\pi k$, and is -1 when $x = \pi \pm 2\pi k$. Similarly, the $-\sin(y)$ term is -1 when $y = \pi/2 + 2\pi k$ and is $+1$ when $y = 3\pi/3 + 2\pi k$.

   iii. See the middle figure below.

   (c) $\begin{aligned} x' &= (2 + x)(y - x) \\ y' &= (4 - x)(y + x) \end{aligned}$

   i. The equilibria are $(-2, 2)$, $(0, 0)$ and $(4, 4)$.

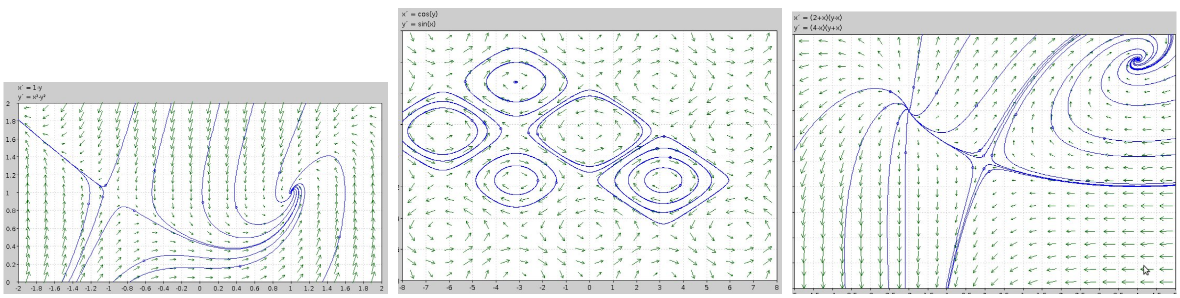
ii. The Jacobian matrix of first partial derivatives is: $\begin{bmatrix} -2+y-2x & 2+x \\ 4-y-2x & 4-x \end{bmatrix}$

At each of the equilibria (in same order as above):

$$\begin{bmatrix} 4 & 0 \\ 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} -2 & 2 \\ 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} -6 & 6 \\ -8 & 0 \end{bmatrix}$$

These are (in order), a source, a saddle and a spiral sink.

iii. See figure below.



2. Consider the following three systems, and label them as A, B, and C:

A. $\begin{aligned} x' &= x\left(\tfrac{3}{2} - x - \tfrac{1}{2}y\right) \\ y' &= y\left(2 - y - \tfrac{3}{4}x\right) \end{aligned}$ B. $\begin{aligned} x' &= x\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right) \\ y' &= y\left(-\tfrac{1}{4} + \tfrac{1}{2}x\right) \end{aligned}$ C. $\begin{aligned} x' &= x(1 - x - y) \\ y' &= y\left(\tfrac{3}{2} - y - x\right) \end{aligned}$

(a) Which systems correspond to a system of competing species? Which correspond to a predator-prey system? (Be sure to explain your reasoning).

SOLUTION: Systems A and C correspond to competing species models: In the absence of the other, each has a logistic model of population growth. In A, C, both populations decrease at a rate proportional to the number of interactions of the two populations, which suggests a competition of resources.

On the other hand, in System B, x is the prey and y is the predator, since the predator population increases proportionally to the number of interactions and the prey population x decreases. In the absence of the other, we also see that x has a logistic model of growth, while y has an exponential *decline*.

(b) In each of the competing species models, determine whether or not there will be peaceful coexistence or extinction. Hint: You only need to determine one equilibrium solution for each.

SOLUTION: We solved similar systems in class. The only equilibrium we need to check for coexistence vs extinction is the equilibrium in the first quadrant $x > 0, y > 0$).

- In System A, this point is the intersection of these two lines, which could be computed using Cramer's Rule:

$$\begin{array}{ll} x+\frac{1}{2}y & =\frac{3}{2} \\ \frac{3}{4}x+y & =2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x & =4/5 \\ y & =7/5 \end{array}$$

  From the Jacobian matrix, insert this point (some fractions involved, but not too bad). An analysis using the Poincaré Diagram shows that the point is a SINK.

  Actually, in System C, the two lines are parallel (so there is no point of intersection). This probably finishes the analysis for C- Since there is no non-zero equilibrium solution for both x and y in System C, one population will almost always go extinct (a graphical analysis would show that x will always go extinct if $y > 0$).

(c) In the predator-prey model, do a complete analysis: Find all equilibria, linearize the system, then use the Poincaré Diagram to classify the equilibrium. Give a conclusion about the corresponding ecosystem- Will there be problems?

SOLUTION: The equilibria are $(0,0)$, $(2,0)$ and $(1/2,3/2)$. The Jacobian matrix is:

$$\left[\begin{array}{cc} 1-x-\frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{2}y & -\frac{1}{4}+\frac{1}{2}x \end{array}\right]$$

We should find that $(0,0)$ and $(2,0)$ are both saddles, and $(1/2,3/2)$ is a spiral sink.

Overview of the ecosystem: This is a very stable system, where the populations will oscillate about $x=1/2$, $y=3/2$.

3. Sometimes nonlinear differential equations can be solved by means of methods from Chapter 2, where we convert the system using $dx/dt, dy/dt$ to dy/dx (we've already done a few of these). Here are some more (one linear one just for fun), as a start to your review for the final! Remember, each DE may be classified in multiple ways, so if you get stuck using one technique, you might try another.

(a) Exact or separable: $\begin{array}{ll} x' & =2y-2 \\ y' & =-(2x+3) \end{array}$ SOLN: $y^2-2y=-x^2-3x+C$

(b) Separable: $\begin{array}{ll} x' & =y(1-x^3) \\ y' & =x^2 \end{array}$ SOLN: $\frac{1}{2}y^2=-\frac{1}{3}\ln|1-x^3|+C$

(c) Homogeneous: $\begin{array}{ll} x' & =x-y \\ y' & =y-4x \end{array} \Rightarrow xv'+v=\dfrac{v^2-4}{1-v}$

Solving, we get:

$$\frac{1-v}{v^2-4}\,dv=\frac{1}{x}\,dx$$

Use partial fractions to integrate the expression on the left, and:

$$\int -\frac{1}{4}\frac{1}{v+2} - \frac{3}{4}\frac{1}{v-2}\,dv = \int \frac{1}{x}\,dx$$

We could write the answer as:

$$\left(\frac{y}{x}+2\right)\left(\frac{y}{x}-2\right)^3 = A\mathbf{x}^{-4}$$

(d) Exact: $\begin{array}{ll} x' & = \mathrm{e}^x\cos(y) + 2\cos(x) \\ y' & = 2y\sin(x) - \mathrm{e}^x\sin(y) \end{array}$

Putting this in standard form,

$$(\mathrm{e}^x\sin(y) - 2y\sin(x)) + (\mathrm{e}^x\cos(y) - 2\cos(x))\frac{dy}{dx} = 0$$

Then

$$M_y = \mathrm{e}^x\cos(y) - 2\sin(x) = N_x$$

And solving for y implicitly as a function of x:

$$\mathrm{e}^x\sin(y) + 2y\cos(x) = C$$

HINT: A typical error is to not finish the answer as an implicit function- That is, don't leave off the "C".

4. (Optional) Here is an interesting system of equations:

$$\begin{array}{ll} x' & = x + y - x(x^2+y^2) \\ y' & = -x + y - y(x^2+y^2) \end{array}$$

Go online to the phase plane plotting website and put these equations in. Draw several solutions to the differential equation, and you should see something very familiar- Something you've probably been looking at all semester long!