**Local Linear Analysis of Nonlinear Autonomous DEs**

Local linear analysis is the process by which we analyze a nonlinear system of differential equations about its equilibrium solutions (also known as critical points or fixed points). Given a system of nonlinear autonomous DEs:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

we first find the equilibrium solutions by setting the derivatives to zero, then solve simultaneously, the system:

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}$$

Given an equilibrium, say $x = a, y = b$, the linearization of the system at the point $(a, b)$ is:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

We can then use the Poincaré Diagram to determine the local behavior. We must use some caution in the case of centers and degenerate nodes, however. Because the linearization is an *approximation* of the true solution, the actual solutions are of a slightly perturbed system. This means that while the linearization gives a center, the true solution may be a center or a spiral (we would use a computer simulation to see what we actually get).

## Example 1: Competing Species

Suppose we have two populations that are competing for similar resources, like rabbits $(x(t))$ and hamsters $(y(t))$.
It seems reasonable to suppose that:

> In the absence of the other, each population is modeled by a population model with an environmental threshold (what we called the **logistic model**. Back in Chapter 2, that was (book's notation on p 81):
>
> $$y' = r(1 - y/k)y = ay - by^2$$
>
> so that $y = 0$ is unstable and $y = k$ is stable.

To simplify matters, we assume some constants:

$$\begin{aligned} x' &= x - x^2 \\ y' &= \tfrac{3}{4}y - y^2 \end{aligned}$$

Now, our second assumption will be:

The rate of change of populations (down) will be proportional to the number of interactions between the populations. For example, if there are 3 rabbits and 2 hamsters, there are a total of 6 possible rabbit-hamster interactions possible.

Our equations become:

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy \end{aligned}$$

The analysis proceeds by getting the **equilibria** (a.k.a. **critical points**):
From the first equation, either $x = 0$ or $x = -y + 1$:

- $x = 0$ in the second equation: $y\left(\frac{3}{4} - y\right) = 0$ so that $y = 0$ or $y = 3/4$.
- If $x = -y + 1$ in the first, then the second equation becomes:

$$y\left(\frac{3}{4} - y - \frac{1}{2}(-y+1)\right) = y\left(\frac{1}{4} - \frac{1}{2}y\right) = 0$$

  Therefore, $y = 0$ (and $x = 0$, but we've counted that one), or $y = 1/2$ (then $x = 1/2$, too).

We have 4 equilibrium solutions:

$$(0,0),\ (1,0),\ (0,3/4),\ (1/2,1/2)$$

Now we linearize the system about each equilibrium solution to determine its stability. First, the matrix of partial derivatives is:

$$\begin{bmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{bmatrix}$$

Evaluating this at each of the critical points (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & 0.25 \end{bmatrix} \quad \begin{bmatrix} 0.25 & 0 \\ -.375 & -.75 \end{bmatrix} \quad \begin{bmatrix} -.5 & -.5 \\ -.25 & -.5 \end{bmatrix}$$

Using the Poincaré Diagram, we see that the origin is indeed a SOURCE, the equilibria on the $x-$ and $y-$ axes are SADDLES, and the point of intersection of the two lines is a SINK. Putting these together, we can look at the direction field to examine the global behavior. From this we see that if both of the initial populations are not zero, the model predicts that all solutions will tend to the sink at $(1/2, 1/2)$- We might call this **peaceful coexistence**.

**Example 2:**

We now repeat the analysis for a slightly different system:

$$\begin{aligned} x' &= x(1 - x - y) &&= x - x^2 - xy \\ y' &= y(0.5 - 0.25y - 0.75x) &&= 0.5y - 0.25y^2 - 0.75xy \end{aligned}$$

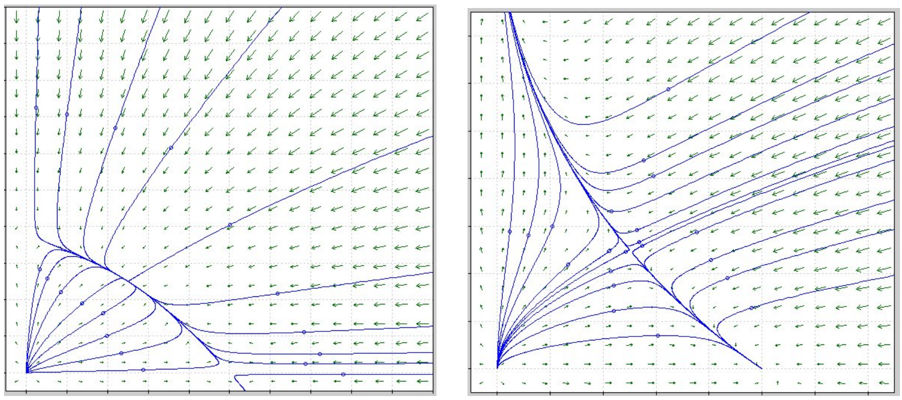

Figure 1: The Competing Species model, with peaceful coexistence. Almost all solutions tend to the equilibrium $(1/2, 1/2)$. In the second case, War!

The critical points have changed slightly:

$$(0,0),\ (1,0),\ (0,2),\ (1/2,1/2)$$

The matrix of partial derivatives is:

$$\begin{bmatrix} 1-2x-y & -x \\ -.75y & 0.5-0.5y-0.75x \end{bmatrix}$$

Evaluating this at each of the critical points (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & -0.25 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ -1.5 & -.5 \end{bmatrix} \quad \begin{bmatrix} -.5 & -.5 \\ -.375 & -.125 \end{bmatrix}$$

Using the Poincaré Diagram, we see that the origin is still a SOURCE, the equilibria on the $x-$ and $y-$ axes are still SADDLES. The big difference is that the point of intersection of the two lines now produces a SADDLE. We will recall that a saddle point in UNSTABLE- this means that for just about any initial condition, one of the two species will die off. Notice how critical these parameters are to the general outcome- This has important public policy implications!

## Predator Prey

The assumptions used here are much like the ones used in the Competing Species model. Here are the general equations (to the left), and our specific example (to the right):

$$\begin{aligned} x' &= ax - bxy &= x(a-by) \qquad & x' &= x - 0.5xy &= x(1-0.5y) \\ y' &= -cy + kxy &= y(-c+kx) \qquad & y' &= -0.75y + 0.25xy &= y(-0.75+0.25x) \end{aligned}$$

Solving for the equilibria, from the first equation we get

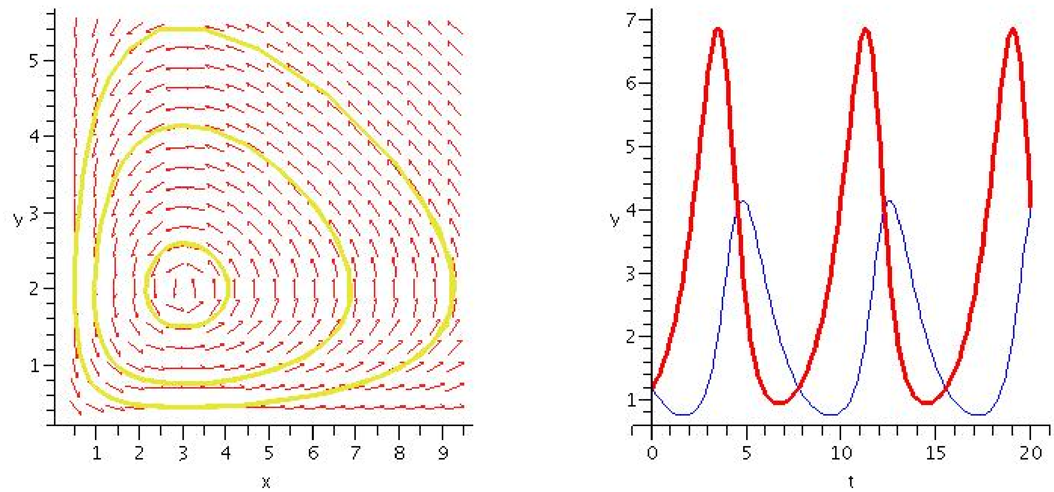


Figure 2: The direction field (top) and a plot of $x(t)$ and $y(t)$ as functions of time (bottom). They both reveal the existence of periodic solutions.

- $x = 0$: From the second, we must have $y = 0$.
- $y = 2$: From the second, $x = 3$.

We have only two equilibria for this system, $(0, 0)$ and $(3, 2)$.
The linearization in general gives:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{bmatrix}$$

Linearizing about the two equilibrium gives (in order):

$$\begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

In the first case, we have a SADDLE at the origin (which is what we could have expected). At the point $(3, 2)$ we have a CENTER. We check the direction field to verify this, and we have also plotted the individual solution, $x(t)$ and $y(t)$ versus $t$.

## The Lorenz Equations

As a fun example of a three dimensional system, consider the Lorenz equations:

$$\begin{aligned} x' &= -10x + 10y \\ y' &= 28x - y - xz \\ z' &= -(8/3)z + xy \end{aligned}$$

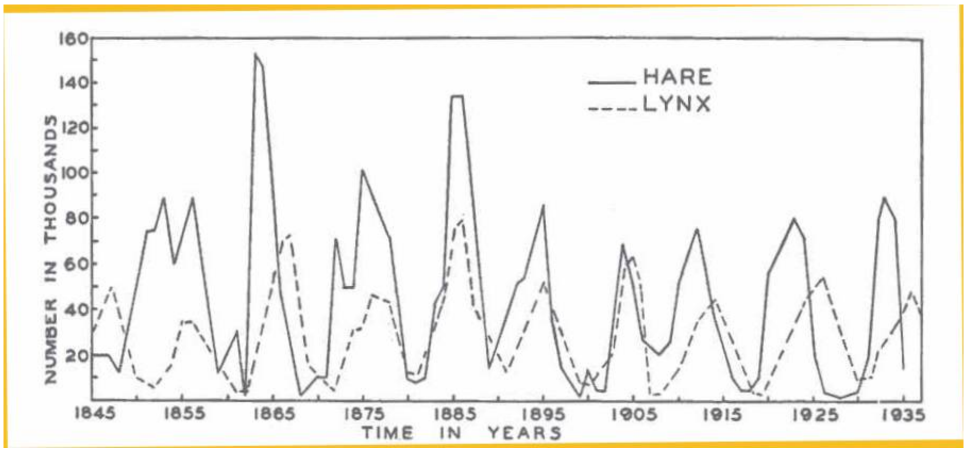


Figure 3: From Odum's "Fundamentals of Ecology". He says that this is taken from an earlier source which is not widely available. Some authors caution that this is actually a composite of several time series, and does not reflect actual populations of rabbits and lynxes, but rather the number of pelts turn in to Hudson's Bay. Is there a predator-prey interaction at work here?

Following our usual mode of analysis, first we find the equilibrium solutions. From the first equation, $x = y$, which we substitute into the last two, giving:

$$\begin{aligned} 0 &= 27x - xz \\ 0 &= -(8/3)z + x^2 \end{aligned}$$

Using the second equation, $x(27 - z) = 0$, then $x = 0$ or $z = 27$. Putting these into the third equation,

- If $x = 0$, then $z = 0$. One equilibrium is $(0, 0, 0)$.
- If $z = 27$, then
  $$x^2 = \frac{216}{3} = 72 \quad x = \pm 6\sqrt{2}$$
  and we have two equilibrium solutions:
  $$(6\sqrt{2}, 6\sqrt{2}, 27) \qquad (-6\sqrt{2}, -6\sqrt{2}, 27)$$

We now linearize the system by examining the matrix of partial derivatives:

$$\begin{bmatrix} -10 & 10 & 0 \\ 28 - z & -1 & -x \\ y & x & -8/3 \end{bmatrix}$$

At the origin, we compute the matrix corresponding to the local linear system. To help us analyze the behavior, we will also include the eigenvalues/vectors of the matrix:

$$\begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}$$

$$\lambda_1 = 11.8, \quad \mathbf{v}_1 = \begin{bmatrix} .46 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_2 = -22.9, \quad \mathbf{v}_2 = \begin{bmatrix} -.78 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -8/3 \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We interpret this as saying that, in the $x-y$ plane, there is a saddle- One direction is attracting, the other is repelling. There is also a downward attracting force in the $z-$component. Notice that in a three-dimensional saddle, we can have several different ways of mixing attracting and repelling directions.

At the other two equilibria, we have:

$$\begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -8/3 \end{bmatrix} \qquad \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & 6\sqrt{2} \\ -6\sqrt{2} & -6\sqrt{2} & -8/3 \end{bmatrix}$$

In both cases, we have the following eigenvalues (I've left off the eigenvectors- We might see them in the direction field):

$$\lambda_1 = -13.85457791, \quad \lambda_{2,3} = .094 \pm 10.19\, i$$

We interpret this to mean that there is one attracting direction, and a (weak) spiral source at each of these equilibria.

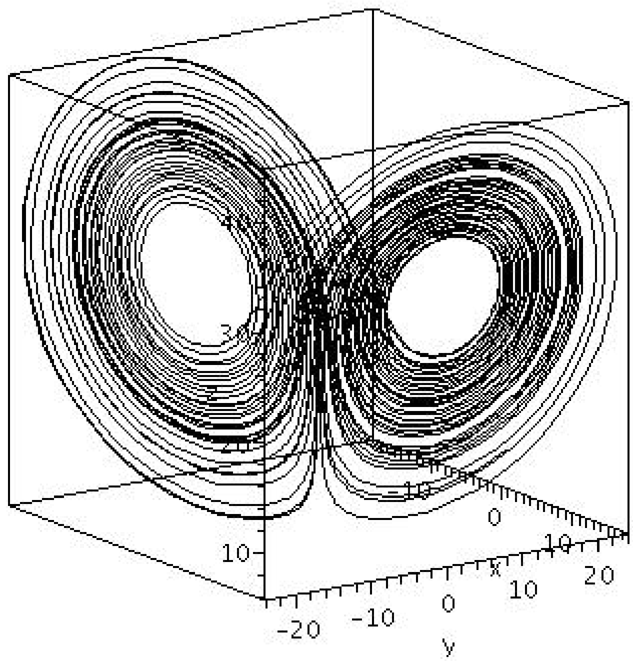


Figure 4: A computed solution to the Lorenz Equations. Notice the appearance of the two weak spiral sources. If you were to run another simulation from an arbitrary starting point, you would get something very similar to this (once the solution got close to this region, and if you don't start at an equilibrium!).