Quick Overview: Complex Numbers

February 23, 2012

1 Initial Definitions

Definition 1 *The* **complex number** z *is defined as:*

$$z = a + bi \tag{1}$$

where a, b *are real numbers and* $i = \sqrt{-1}$.

Remarks about the definition:

- Engineers typically use j instead of i.
- Examples of complex numbers:

$$5 + 2i, \quad 3 - \sqrt{2}i, \quad 3, \quad -5i$$

- Powers of i:

$$\begin{array}{rr} i^2 = -1 & i^3 = -i \\ i^4 = 1 & i^5 = i \\ i^6 = -1 & i^7 = -i \\ & \vdots \end{array}$$

- All real numbers are also complex (by taking $b = 0$).

2 Visualizing Complex Numbers

A complex number is defined by it's two real numbers. If we have $z = a + bi$, then:

Definition 2 *The* **real part** *of* $a + bi$ *is* a,

$$\mathrm{Re}(z) = \mathrm{Re}(a + bi) = a$$

The **imaginary part** *of* $a + bi$ *is* b,

$$\mathrm{Im}(z) = \mathrm{Im}(a + bi) = b$$

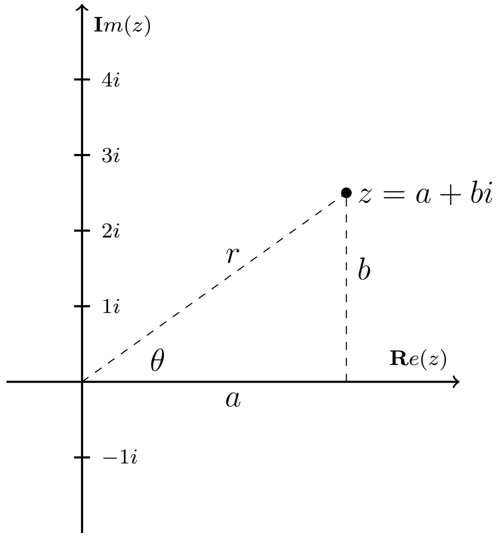

Figure 1: Visualizing $z = a + bi$ in the complex plane. Shown are the modulus (or length) r and the argument (or angle) θ.

To visualize a complex number, we use the complex plane $\mathbb{C}$, where the horizontal (or x-) axis is for the real part, and the vertical axis is for the imaginary part. That is, $a + bi$ is plotted as the point (a, b).

In Figure 1, we can see that it is also possible to represent the point $a + bi$, or (a, b) in polar form, by computing its modulus (or size), and angle (or argument):

$$r = |z| = \sqrt{a^2 + b^2} \qquad \theta = \arg(z)$$

We have to be a bit careful defining ϕ, since there are many ways to write ϕ (and we could add multiples of 2π as well). Typically, the argument of the complex number $z = a + bi$ is defined to be the 4-quadrant "inverse tangent"[1] that returns $-\pi < \theta \leq \pi$.

That is, formally we can define the argument as:

$$\theta = \arg(a + bi) = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0 & \text{(Quad I and IV)} \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0 & \text{(Quad II)} \\ \tan^{-1}(b/a) - \pi & \text{if } a < 0 \text{ and } b < 0 & \text{(Quad III)} \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 & \text{(Upper imag axis)} \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 & \text{(Lower imag axis)} \\ \text{Undefined} & \text{if } x = 0 \text{ and } y = 0 & \text{(The origin)} \end{cases}$$

Examples

Find the modulus r and argument θ for the following numbers (Hint: It is easiest to visualize these in the plane):

- $z = -3$: SOLUTION: $r = 3$ and $\theta = \pi$
- $z = 2i$: SOLUTION: $r = 2$ and $\theta = \pi/2$

[1] For example, in Maple this special angle is computed as `arctan(b,a)`, and in Matlab the command is `atan2(b,a)`.

- $z = -1 + i$: SOLUTION: $r = \sqrt{2}$ and $\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$
- $z = -3 - 2i$ (Numerical approx from Calculator OK):
  SOLUTION: $r = \sqrt{14}$ and $\theta = \tan^{-1}(2/3) - \pi \approx 0.588 - \pi \approx -2.55$ rad

3 Operations on Complex Numbers

3.1 The Conjugate of a Complex Number

If $z = a + bi$ is a complex number, then its *conjugate*, denoted by $\bar{z}$ is $a - bi$. For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i \qquad z = i \Rightarrow \bar{z} = -i \qquad z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is it's mirror image across the horizontal axis. If z has magnitude r and argument θ, then $\bar{z}$ has the same magnitude with a negative argument.

3.2 Addition/Subtraction, Multiplication/Division

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately:

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

To multiply, expand it as if you were multiplying polynomials:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

and simplify using $i^2 = -1$. Note what happens when you multiply a number by its conjugate:

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = |z|^2$$

Division by complex numbers z, w: $\frac{z}{w}$, is defined by translating it to real number division (rationalize the denominator):

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Example:

$$\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{34} = \frac{-7}{34} + \frac{11}{34}i$$

4 The Polar Form of Complex Numbers

4.1 Euler's Formula

Any point on the unit circle can be written as $(\cos(\theta), \sin(\theta))$, which corresponds to the complex number $\cos(\theta) + i\sin(\theta)$. It is possible to show the following directly, but we'll use it as a definition:

Definition (Euler's Formula): $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

4.2 Polar Form of $a+bi$:

The polar form is defined as:

$$z = re^{i\theta} \quad \text{where} \quad r = |z| = \sqrt{a^2+b^2} \qquad \theta = \arg(z)$$

4.2.1 Examples

Given the complex number in $a+bi$ form, give the polar form, and vice-versa:

1. $z = 2i$ SOLUTION: Since $r = 2$ and $\theta = \pi/2$, $z = 2e^{i\pi/2}$
2. $z = 2e^{-i\pi/3}$
   We recall that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, so
   $$z = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)) = 1 - \sqrt{3}\,i$$

5 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form:

$$\ln(a+bi) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta$$

The logarithm of zero is left undefined (as in the real case). However, we can now compute the log of a negative number:

$$\ln(-1) = \ln\left(1 \cdot e^{i\pi}\right) = i\pi \qquad \text{or the log of } i: \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

Note that the usual rules of exponentiation and logarithms still apply.

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like x^x):

$$a^b = e^{b\ln(a)}$$

Example, $2^i = e^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$
Example: $\sqrt{1+i} = (1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = (2^{1/4})e^{i\pi/8}$
Example: $i^i = e^{i\ln(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$

6 Real Polynomials and Complex Numbers

If $ax^2+bx+c=0$, then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2+1=0$ are $x=i$ and $x=-i$.

Check:

$$(x-i)(x+i) = x^2 + xi - xi - i^2 = x^2+1$$

Some facts about polynomials when we allow complex roots:

1. An n^{th} degree polynomial can always be factored into n roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.

2. If $a+bi$ is a root to a real polynomial, then $a-bi$ must also be a root. This is sometimes referred to as "roots must come in conjugate pairs".

7 Exercises

1. Suppose the roots to a cubic polynomial are $a=3$, $b=1-2i$ and $c=1+2i$. Compute $(x-a)(x-b)(x-c)$.

2. Find the roots to $x^2-2x+10$. Write them in polar form.

3. Show that:
$$\text{Re}(z)=\frac{z+\bar{z}}{2} \qquad \text{Im}(z)=\frac{z-\bar{z}}{2i}$$

4. For the following, let $z_1=-3+2i$, $z_2=-4i$
   (a) Compute $z_1\bar{z}_2$, z_2/z_1
   (b) Write z_1 and z_2 in polar form.

5. In each problem, rewrite each of the following in the form $a+bi$:
   (a) e^{1+2i}
   (b) e^{2-3i}
   (c) $\mathrm{e}^{i\pi}$
   (d) 2^{1-i}
   (e) $\mathrm{e}^{2-\frac{\pi}{2}i}$
   (f) π^i

6. For fun, compute the logarithm of each number:
   (a) $\ln(-3)$
   (b) $\ln(-1+i)$
   (c) $\ln(2\mathrm{e}^{3i})$