Complex Integrals and the Laplace Transform

There are a few computations for which the complex exponential is very nice to use. We'll see a few here, but first a couple of Theorems about integrating a complex function:

Theorem: $\int \mathrm{e}^{(bi)t}\,dt = \frac{1}{bi}\mathrm{e}^{(bi)t}$
Proof:

$$\int \mathrm{e}^{(bi)t}\,dt = \int \mathrm{e}^{(bt)i}\,dt = \int \cos(bt) + i\sin(bt)\,dt = \int \cos(bt)\,dt + i\int \sin(bt)\,dt =$$

$$\frac{1}{b}\sin(bt) - \frac{i}{b}\cos(bt) = \frac{\sin(bt) - i\cos(bt)}{b}$$

And

$$\frac{1}{bi}\mathrm{e}^{(bt)i} = \frac{\cos(bt) + i\sin(bt)}{bi} \cdot \frac{i}{i} = \frac{-\sin(bt) + i\cos(bt)}{-b} = \frac{\sin(bt) - i\cos(bt)}{b}$$

Therefore, these quantities are the same.

Theorem: $\int \mathrm{e}^{(a+bi)t}\,dt = \frac{1}{(a+bi)}\mathrm{e}^{(a+bi)t}$
You can work this out, but it is more complicated since we'll need to do integration by parts twice for each integral. It is a nice exercise to try out when you have a little time.

Theorem: The main computational technique is using the following:

$$\int \mathrm{e}^{at}\cos(bt)\,dt = \mathrm{Re}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Re}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

$$\int \mathrm{e}^{at}\sin(bt)\,dt = \mathrm{Im}\left(\int \mathrm{e}^{(a+bi)t}\,dt\right) = \mathrm{Im}\left(\frac{1}{a+ib}\mathrm{e}^{(a+ib)t}\right)$$

Worked Example:

1. Use complex exponentials to compute $\int \mathrm{e}^{2t}\cos(3t)\,dt$.

   SOLUTION: We note that $\mathrm{e}^{2t}\cos(3t) = \mathrm{Re}(\mathrm{e}^{(2+3i)t})$, so:

   $$\int \mathrm{e}^{2t}\cos(3t)\,dt = \mathrm{Re}\left(\frac{1}{2+3i}\mathrm{e}^{(2+3i)t}\right)$$

   Simplifying the term inside the parentheses and multiplying out the complex terms:

   $$\mathrm{e}^{2t}\left(\frac{2-3i}{4+9}\right)(\cos(3t) + i\sin(3t)) =$$

   $$\mathrm{e}^{2t}\left[\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right) + i\left(-\frac{3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)\right]$$

   Therefore,

   $$\int \mathrm{e}^{2t}\cos(3t)\,dt = \mathrm{e}^{2t}\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right)$$

In fact, we get the other integral for free:

$$\int e^{2t}\sin(3t)\,dt = e^{2t}\left(\frac{-3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)$$

2. Use complex exponentials to compute $\int \sin(at)\,dt$

   This one is simple enough to do without using complex exponentials, but it does still work.

$$\int \sin(at)\,dt = \text{Im}\left(\left(\int e^{(at)i}\,dt\right) = \text{Im}\left(\frac{1}{ai}(\cos(at) + i\sin(at)\right)\right) =$$

$$\text{Im}\left(\frac{-i}{a}(\cos(at) + i\sin(at))\right) = \text{Im}\left(\frac{1}{a}\sin(at) + i\left(\frac{-1}{a}\cos(at)\right)\right) = \frac{-1}{a}\cos(at)$$

3. Use complex exponentials to compute the Laplace transform of $\cos(at)$:

   SOLUTION: Note that $\cos(at) = \text{Re}(e^{(at)i})$

$$\mathcal{L}(\cos(at)) = \int_0^\infty e^{-st}\cos(at)\,dt = \text{Re}\left(\int_0^\infty e^{-st}e^{(ai)t}\,dt\right) =$$

$$\text{Re}\left(\int_0^\infty e^{-(s-ai)t}\,dt\right) = \text{Re}\left(\frac{-1}{(s-ai)}e^{-(s-ai)t}\Big|_{t=0}^{t\to\infty}\right.$$

   What happens to our expression as $t \to \infty$? The easiest way to take the limit is to check the magnitude (see if it is going to zero):

$$\left|\frac{-1}{s-ai}e^{-st}e^{(ai)t}\right| = \left|\frac{-1}{s-ai}\right| \cdot \left|e^{-st}\right| \cdot \left|e^{(ai)t}\right|$$

   Now, the first term is a constant and $e^{(at)i}$ is a point on the unit circle (so its magnitude is 1). Therefore, the magnitude depends solely on e^{-st}, where s is any real number.

   And, the function $e^{-st} \to 0$ as $t \to \infty$ for any $s > 0$. Therefore,

$$\lim_{t\to\infty}\frac{-1}{(s-ai)}e^{-(s-ai)t} = 0$$

   and the Laplace transform is:

$$\mathcal{L}(\cos(at)) = \text{Re}\left(0 - \frac{-1}{s-ai}\right) = \text{Re}\left(\frac{s+ai}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

   As a side remark, we get the Laplace transform of $\sin(at)$ for free since it is the imaginary part.

Homework Addition to Section 6.1

1. Use complex exponentials to compute $\int e^{-2t}\sin(3t)\,dt$.

2. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

3. Use complex exponentials to compute the Laplace transform of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$ (compare to exercises 13, 14).

4. Show that, if $f(t)$ is bounded (that is, there is a constant A so that $|f(t)| \leq A$ for all t), then f is of exponential order (do this by finding K, a and M from the definition).

5. If the function is of exponential order, find the K, a and M from the definition. Otherwise, state that it is not of exponential order.

   Something that may be handy from algebra: $A = e^{\ln(A)}$.

   (a) $\sin(t)$

   (b) $\tan(t)$

   (c) t^3

   (d) e^{t^2}

   (e) 5^t

   (f) t^t

6. Use complex exponentials to find the Laplace transform of $t\sin(at)$.

Homework Addition Solutions

1. Use complex exponentials to compute $\int e^{-2t}\sin(3t)\,dt$.

   SOLUTION:

$$\int e^{-2t}\sin(3t)\,dt = \int \mathrm{Im}\left(e^{(-2+3i)t}\,dt\right) = \mathrm{Im}\left(\frac{1}{-2+3i}e^{(-2+3i)t}\right) =$$

$$\mathrm{Im}\left(\left[-\frac{2}{13}-\frac{3}{13}i\right]\cdot(e^{-2t}\cos(3t)+ie^{-2t}\sin(3t)\right) = -\frac{3}{13}e^{-2t}\cos(3t)-\frac{2}{13}e^{-2t}\sin(3t)$$

2. Use complex exponentials to compute the Laplace transform of $\sin(at)$.

   SOLUTION:

$$\mathcal{L}(\sin(at)) = \int_0^\infty e^{-st}\sin(at)\,dt$$

   Ignoring the bounds for a bit,

$$\mathrm{Im}\left(\int e^{(-s+ai)t}\,dt\right) = \mathrm{Im}\left(\frac{1}{-s+ai}e^{(-s+ai)t}\right) =$$

$$\mathrm{Im}\left(\left[-\frac{s}{s^2+a^2}-\frac{a}{s^2+a^2}i\right]\cdot(e^{(-s+ai)t}\Big|_{t=0}^{t\to\infty}\right.$$

   As we showed earlier, if $t\to\infty$, then

$$e^{-(s-ai)t} = e^{-st}e^{(at)i}\to 0$$

   as long as $s>0$ (because $|e^{(at)i}|=1$). Therefore,

$$\mathcal{L}(\sin(at)) = 0 - -\frac{a}{s^2+a^2} = \frac{a}{s^2+a^2}$$

3. Use complex exponentials to compute the Laplace transform of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$ (compare to exercises 13, 14).

   SOLUTION: This is very much the same analysis as before, except that

$$\mathcal{L}(e^{at}\cos(bt)) + i\mathcal{L}(e^{at}\sin(bt)) = \mathcal{L}(e^{(a+bi)t}) = \int_0^\infty e^{-st}e^{(a+bi)t}\,dt =$$

$$\int_0^\infty e^{-((s-a)-bi)t}\,dt = \left(-\frac{1}{(s-a)-bi}e^{-((s-a)-bi)t}\right|_0^{t\to\infty}$$

   As $t\to\infty$, the exponential term will go to zero as long as $s-a>0$, or $s>a$. If that is true, then we have:

$$= \frac{1}{(s-a)-bi} = \frac{s-a}{(s-a)^2+b^2}+\frac{b}{(s-a)^2+b^2}i$$

   From this, we get:

$$\mathcal{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

4. Show that, if $f(t)$ is bounded (that is, there is a constant A so that $|f(t)| \leq A$ for all t), then f is of exponential order (do this by finding K, a and M from the definition).

   SOLUTION: If $f(t)$ is bounded, then

$$|f(t)| \leq A = A \cdot e^{0 \cdot t}$$

   for all t.

5. If the function is of exponential order, find the K, a, M from the definition. Otherwise, state that it is not of exponential order.

   Something that may be handy from algebra: $A = e^{\ln(A)}$.

   (a) $\sin(t)$

   SOLUTION: $\sin(t)$ is bounded by 1, so $K = 1$, $a = 0$, and M is irrelevant (true for all t).

   (b) $\tan(t)$

   SOLUTION: Since the tangent function has vertical asymptotes, $\tan(t)$ is not of exponential order.

   (c) t^3

   SOLUTION: Consider $t > 0$:

$$t^3 = t^3 = e^{\ln(t^3)} = e^{3 \ln t} \leq e^{3t}$$

   Therefore, $K = 1$, $a = 3$ and $M = 0$

   (d) e^{t^2}

   SOLUTION: Not of exponential order, since we're raising t to a polynomial power (larger than 1).

   (e) 5^t

   SOLUTION:

$$5^t = e^{\ln(5^t)} = e^{\ln(5)\, t}$$

   so $K = 1$, $a = \ln(5)$ and $M = 0$.

   (f) t^t

   SOLUTION: t^t is not of exponential order, since $t^t = e^{t \ln(t)}$ and

$$t \ln(t) > at$$

   for any constant a.

6. Use complex exponentials to find the Laplace transform of $t \sin(at)$.

   SOLUTION: Using the definition, we'll consider the imaginary part of the following integral:

$$\int_0^\infty e^{-st}\, t e^{ait}\, dt = \int_0^\infty t e^{-(s-ai)t}\, dt$$

Using integration by parts,

$+$	t	$e^{-(s-ai)t}$
$-$	1	$-1/(s-ai)\,e^{-(s-ai)t}$
$+$	0	$1/(s-ai)^2\,e^{-(s-ai)t}$

$$\Rightarrow \quad e^{-(s-ai)t}\left(-\frac{t}{s-ai}-\frac{1}{(s-ai)^2}\right)$$

The term in the parentheses will go to zero as long as the exponential goes to zero- Which it will as long as $s>0$. In that case, the integral becomes:

$$\frac{1}{(s-ai)^2}=\frac{1}{(s^2-a^2)-2asi}$$

When we multiply by the conjugate, the denominator will become:

$$(s^2-a^2)^2+4a^2s^2=s^4-2a^2s^2+a^4+4a^2s^2=s^4+2a^2s^2+a^4=(s^2+a^2)^2$$

so that finally we get:

$$\frac{1}{(s-a)^2}=\frac{s^2-a^2}{(s^2+a^2)^2}+\frac{2as}{(s^2+a^2)^2}i$$

Therefore, our final answer is the imaginary part of this,

$$\frac{2as}{(s^2+a^2)}$$

For future reference, you might verify that this expression is actually:

$$(-1)\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

which is how we will be computing this transform later...