Review Questions: Exam 3

1. What is the ansatz we use for y in
   - Chapter 6?
   - Section 5.2-5.3?
   - Section 5.4 (for $x^2y'' + \alpha xy' + \beta y = 0$)?

2. Finish the definitions:
   - The Heaviside function, $u_c(t)$:
   - The Dirac $\delta-$function: $\delta(t-c)$ (Note: the Dirac function should be defined as a certain limit)
   - Define the convolution: $(f * g)(t)$
   - A function is of **exponential order** if:

3. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6-t, & t \ge 2 \end{cases}$$

   (b)
   $$f(t) = \begin{cases} e^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

4. Check your answers to Problem 2 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

5. Write the following functions in piecewise form (thus removing the Heaviside function):

   (a) $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$

   (b) $\displaystyle\sum_{n=1}^{4} u_{n\pi}(t)\sin(t-n\pi)$

6. Determine the Laplace transform:

   (a) t^2e^{-9t}

   (b) $e^{2t} - t^3 - \sin(5t)$

   (c) $t^2y'(t)$ (in terms of $Y(s)$)

   (d) $e^{3t}\sin(4t)$

   (e) $e^t\delta(t-3)$

   (f) $t^2u_4(t)$

7. Find the inverse Laplace transform:

   (a) $\dfrac{2s-1}{s^2-4s+6}$

   (b) $\dfrac{7}{(s+3)^3}$

   (c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$

   (d) $\dfrac{3s-1}{2s^2-8s+14}$

   (e) $\left(e^{-2s} - e^{-3s}\right)\dfrac{1}{s^2+s-6}$

8. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0, \quad y'(0) = -1$

   (b) $y'' + 9y = 10\mathrm{e}^{2t}$, $y(0) = -1, \quad y'(0) = 5$

   (c) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0, \quad y'(0) = 0$

9. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3, \quad y'(0) = 10$

   (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0, \quad y'(0) = -1$

   (d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0, \quad y'(0) = 1$

   (e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

10. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

    (a) $4y'' + 4y' + 17y = g(t) \qquad y(0) = 0, y'(0) = 0$

    (b) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$, with $y(0) = 1$ and $y'(0) = -1$.

11. Short Answer:

    (a) $\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt =$ ____________________

    (b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

    $$y'(t) = \delta(t-c), \qquad y(0) = 0$$

    (c) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2 + 2x + 5)$ if the series is based at $x_0 = 1$?

    (d) Use Laplace transforms to solve for $F(s)$, if

    $$f(t) + 2\int_0^t \cos(t-x) f(x)\, dx = \mathrm{e}^{-t}$$

    (So only solve for the transform of $f(t)$, don't invert it back).

    (e) In order for the Laplace transform of f to exist, f must be ____________________

    (f) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

    (g) Use the table to find the Laplace transform of $\mathrm{e}^{-2t}\sinh(\sqrt{3}\,t)$. (Note: You don't need the definition the hyperbolic sine to answer this question).

    (h) Is $x = 0$ an ordinary point for the differential equation: $xy'' + 3x^2y' + y = 4$?

12. Let $f(t) = t$ and $g(t) = u_2(t)$.

(a) Use the Laplace transform to compute $f * g$.

(b) Verify your answer by computing $f * g$ using the definition.

13. If $a_0 = 1$, determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

14. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k (x-1)^k$ (with a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1) a_n (x-1)^{n-2} + x(x-2) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

15. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \qquad y(0) = 1, y'(0) = 1$$

16. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

17. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

(b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$

(c) $y'' - xy' - y = 0$, $x_0 = 1$

18. Exercises with the table:

(a) Use table entries 5 and 14 to prove the formula for 9.

(b) Show that you can use table entry 19 to find the Laplace transform of $t^2 \delta(t-3)$ (verify your answer using a property of the δ function).

(c) Prove (using the definition of $\mathcal{L}$) table entries 12 and 13.

(d) Prove (using the definition of $\mathcal{L}$) a formula (similar to 18) for $\mathcal{L}(y'''(t))$.

19. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

20. Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation:

$$(x^2 - 2x + 5)y'' + 4xy' + y = 0 \qquad x_0 = 0, \qquad x_0 = 3$$

21. Find the radius of convergence for all of the following, and find the interval of convergence for (b) and (d):

(a) $\sum_{n=1}^{\infty} \sqrt{n}x^n$

(c) $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}$ (A little tricky)

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

(d) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$

22. Exercises from 5.4, as assigned and if applicable.