Solutions to Review Questions: Exam 3

1. What is the ansatz we use for y in

   - Chapter 6? SOLUTION: $y(t)$ is piecewise continuous and is of exponential order (so that $Y(s)$ exists).
   - Section 5.2-5.3? SOLUTION: $y(x)$ is analytic at $x = x_0$. That is,

     $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

   - Section 5.4 (for $x^2y'' + \alpha xy' + \beta y = 0$)? SOLUTION: $y = x^r$ (and it is good to recall that the substitution $t = \ln(x)$ connects this problem to $\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0$, which we solved in Chapter 3).

2. Finish the definitions:

   - The Heaviside function, $u_c(t)$:

     $$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \qquad c > 0$$

   - The Dirac $\delta-$function: $\delta(t - c)$

     $$\delta(t - c) = \lim_{\tau \to 0} d_\tau(t - c)$$

     where

     $$d_\tau(t - c) = \begin{cases} \frac{1}{2\tau} & \text{if } c - \tau < t < c + \tau \\ 0 & \text{elsewhere} \end{cases}$$

   - Define the convolution: $(f * g)(t)$

     $$(f * g)(t) = \int_0^t f(t - u) g(u)\, du$$

   - A function is of **exponential order** if:
     there are constants M, k, and a so that

     $$|f(t)| \leq M e^{kt} \quad \text{for all} \quad t \geq a$$

3. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)

   $$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

   $$\int_0^\infty e^{-st} f(t)\, dt = \int_0^2 3e^{-st}\, dt + \int_2^\infty (6 - t) e^{-st}\, dt$$

The second antiderivative is found by integration by parts:

$$\int_2^\infty (6-t)e^{-st}\,dt \Rightarrow \quad \begin{array}{ccc} \hline + & 6-t & e^{-st} \\ - & -1 & (-1/s)e^{-st} \\ + & 0 & (1/s^2)e^{-st} \end{array} \quad \Rightarrow \quad e^{-st}\left(-\frac{6-t}{s}+\frac{1}{s^2}\right)\Bigg|_2^\infty$$

Putting it all together,

$$-\frac{3}{s}e^{-st}\Big|_0^2+\left(0-e^{-2s}\left(-\frac{4}{s}+\frac{1}{s^2}\right)\right)=-\frac{3e^{-2s}}{s}+\frac{3}{s}+\frac{4e^{-2s}}{s}-\frac{e^{-2s}}{s^2}=\frac{3}{s}+e^{-2s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

NOTE: Did you remember to *anti*differentiate in the third column?

(b)

$$f(t)=\begin{cases} e^{-t}, & 0\le t<5 \\ -1, & t\ge 5\end{cases}$$

$$\int_0^\infty e^{-st}f(t)\,dt=\int_0^5 e^{-st}e^{-t}\,dt+\int_5^\infty -e^{-st}\,dt=\int_0^5 e^{-(s+1)t}\,dt+\int_5^\infty -e^{-st}\,dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}e^{-(s+1)t}\Big|_0^5+\frac{1}{s}e^{-st}\Big|_5^\infty=\frac{1}{s+1}-\frac{e^{-5(s+1)}}{s+1}+0-\frac{e^{-5s}}{s}$$

4. Check your answers to Problem 2 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

   (a) $f(t)=3(u_0(t)-u_2(t))+(6-t)u_2(t)=3-3u_2(t)+(6-t)u_2(t)=3+(3-t)\,u_2(t)$
   For the second term, notice that the table entry is for $u_c(t)h(t-c)$. Therefore, if

   $$h(t-2)=3-t \quad \text{then} \quad h(t)=3-(t+2)=1-t \quad \text{and} \quad H(s)=\frac{1}{s}-\frac{1}{s^2}$$

   Therefore, the overall transform is:

   $$\frac{3}{s}+e^{-2s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$$

   (b) $f(t)=e^{-t}\,(u_0(t)-u_5(t))-u_5(t)$
   We can rewrite f in preparation for the transform:

   $$f(t)=e^{-t}u_0(t)-e^{-t}u_5(t)-u_5(t)$$

   For the middle term,

   $$h(t-5)=e^{-t} \quad \Rightarrow \quad h(t)=e^{-(t+5)}=e^{-5}e^{-t}$$

   so the overall transform is:

   $$F(s)=\frac{1}{s+1}-e^{-5}\frac{e^{-5s}}{s+1}-\frac{e^{-5s}}{s}$$

5. Write the following functions in piecewise form (thus removing the Heaviside function):

(a) $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$
SOLUTION: First, notice that $(t+2)$ is turned "on" at time 2. At time $t = 3$, $\sin(t)$ joins the first function, and at time $t = 4$, we subtract the function $t+2$ back off.

$$\begin{cases} 0 & \text{if } 0 \le t < 2 \\ t+2 & \text{if } 2 \le t < 3 \\ t+2+\sin(t) & \text{if } 3 \le t < 4 \\ \sin(t) & \text{if } t \ge 4 \end{cases}$$

(b) $\displaystyle\sum_{n=1}^{4} u_{n\pi}(t)\sin(t - n\pi)$
SOLUTION: First (you can determine this graphically) $\sin(t-\pi) = -\sin(t)$, and $\sin(t-2\pi) = \sin(t)$, and $\sin(t-3\pi) = -\sin(t)$, etc.- You should simplify these. Therefore:

$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ \sin(t-\pi) & \text{if } \pi \le t < 2\pi \\ \sin(t-\pi)+\sin(t-2\pi) & \text{if } 2\pi < t < 3\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi) & \text{if } 3\pi \le t < 4\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi)+\sin(t-4\pi) & \text{if } t \ge 4\pi \end{cases} =$$

$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi < t < 3\pi \\ -\sin(t) & \text{if } 3\pi \le t < 4\pi \\ 0 & \text{if } t \ge 4\pi \end{cases}$$

6. Determine the Laplace transform:

(a) $t^2 e^{-9t}$

$$\frac{2}{(s+9)^3}$$

(b) $e^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

(c) $t^2 y'(t)$. Use Table Entry 16, $\mathcal{L}(-t^n f(t)) = F^{(n)}(s)$. In this case, $F(s) = sY(s) - y(0)$, so $F'(s) = sY'(s) + Y(s)$ and $F''(s) = sY''(s) + 2Y'(s)$.

(d) $e^{3t}\sin(4t)$

$$\frac{4}{(s-3)^2+16}$$

(e) $e^t \delta(t-3)$
In this case, notice that $f(t)\delta(t-c)$ is the same as $f(c)\delta(t-c)$, since the delta function is zero everywhere except at $t = c$. Therefore,

$$\mathcal{L}(e^t\delta(t-c)) = e^3 e^{-3s}$$

(f) $t^2u_4(t)$

In this case, let $h(t-4)=t^2$, so that

$$h(t)=(t+4)^2=t^2+8t+16 \qquad \Rightarrow \qquad H(s)=\frac{2+8s+16s^2}{s^3}$$

and the overall transform is $\mathrm{e}^{-4s}H(s)$.

7. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6}=\frac{2s-1}{(s^2-4s+4)+2}=2\frac{s-1/2}{(s-2)^2+2}=$$

In the numerator, make $s-\frac{1}{2}$ into $s-2+\frac{3}{2}$, then

$$2\left(\frac{s-2}{(s-2)^2+2}+\frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s-2)^2+2}\right)\Rightarrow 2\mathrm{e}^{2t}\cos(\sqrt{2}t)+\frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin(\sqrt{2}t)$$

(b) $\dfrac{7}{(s+3)^3}=\dfrac{7}{2!}\dfrac{2!}{(s+3)^3}\Rightarrow\dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}=\mathrm{e}^{-2s}H(s)$, where

$$H(s)=\frac{4s+2}{(s-1)(s+2)}=\frac{2}{s-1}+\frac{2}{s+2} \qquad \Rightarrow \qquad h(t)=2\mathrm{e}^{t}+2\mathrm{e}^{-2t}$$

and the overall inverse: $u_2(t)h(t-2)$.

(d) $\dfrac{3s-1}{2s^2-8s+14}$ Complete the square in the denominator, factoring the constants out:

$$\frac{3s-1}{2(s^2-8s+5)}=\frac{3}{2}\cdot\frac{s-1/3}{(s-2)^2+3}=\frac{3}{2}\left(\frac{s-2}{(s-2)^2+3}+\frac{5}{3}\cdot\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-2)^2+3}\right)$$

The inverse transform is:

$$\frac{3}{2}\mathrm{e}^{2t}\cos(\sqrt{3}t)+\frac{5}{2\sqrt{3}}\mathrm{e}^{2t}\sin(\sqrt{3}t)$$

(e) $\left(\mathrm{e}^{-2s}-\mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6}=\left(\mathrm{e}^{-2s}-\mathrm{e}^{-3s}\right)H(s)$

Where:

$$H(s)=\frac{1}{s^2+s-6}=\frac{1}{5}\frac{1}{s-2}-\frac{1}{5}\frac{1}{s+3}$$

so that

$$h(t)=\frac{1}{5}\mathrm{e}^{2t}-\frac{1}{5}\mathrm{e}^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2)-u_3(t)h(t-3)$$

8. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

   so that

$$Y(s) = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}$$

   (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$

$$s^2Y + s - 5 + 9Y = \frac{10}{s-2} \Rightarrow Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9}$$

   (c) $y'' - 4y' + 4y = t^2e^t$, $y(0) = 0$, $y'(0) = 0$

$$(s^2 - 4s + 4)Y = \frac{2}{(s-1)^3} \Rightarrow Y(s) = \frac{2}{(s-1)^3(s-2)^2}$$

9. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

$$Y = \frac{e^{-5s}}{2s^2+s+2} = e^{-5s}H(s)$$

   where

$$H(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\frac{1}{s^2+\frac{1}{2}s+1} = \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

   Therefore,

$$h(t) = \frac{2}{\sqrt{15}}e^{-1/4\,t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

   And the overall solution is $u_5(t)h(t-5)$

   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

$$s^2Y + 3s - 10 + 6(sY+3) + 9Y = 0 \quad \Rightarrow \quad Y = -\frac{3s+8}{(s+3)^2}$$

   Partial Fractions:

$$-\frac{3s+8}{(s+3)^2} = -\frac{3}{(s+3)} + \frac{1}{(s+3)^2} \Rightarrow y(t) = -3e^{-3t} + te^{-3t}$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

$$Y = \mathrm{e}^{-s}\frac{1}{s(s-3)(s+1)} + \frac{1}{(s+1)(s-3)} = \mathrm{e}^{-s}H(s) + \frac{1}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3}\frac{1}{s} + \frac{1}{12}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}\mathrm{e}^{3t} + \frac{1}{4}\mathrm{e}^{-t}$$

and the overall solution is:

$$y(t) = \frac{1}{4}\mathrm{e}^{3t} - \frac{1}{4}\mathrm{e}^{-t} + u_1(t)h(t-1)$$

(d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

$$Y = \mathrm{e}^{-\pi/2\, s}\frac{1}{s^2+4} + \frac{1}{s^2+4}$$

Therefore,

$$y(t) = \frac{1}{2}\sin(2t) + u_{\pi/2}(t)\frac{1}{2}\sin(2(t-\pi/2))$$

(e) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty} \mathrm{e}^{-2k\pi s}\frac{1}{s^2+1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty} u_{2k\pi}(t)\sin(t - 2\pi k) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2\pi \\ \sin(t) & \text{if} \quad 2\pi \le t < 4\pi \\ 2\sin(t) & \text{if} \quad 4\pi \le t < 6\pi \\ 3\sin(t) & \text{if} \quad 6\pi \le t < 8\pi \\ \vdots & \vdots \end{cases}$$

10. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

(a) $4y'' + 4y' + 17y = g(t) \qquad y(0) = 0, y'(0) = 0$
SOLUTION: First, note that

$$4s^2 + 4s + 17 = 4(s^2 + s + 17/4) = 4((s+1/2)^2 + 4)$$

Therefore,

$$Y(s) = \frac{G(s)}{4s^2+4s+17} = G(s)\cdot\frac{1}{8}\frac{2}{(s+\frac{1}{2})^2+2^2}$$

Therefore,

$$y(t) = g(t) * \frac{1}{8}\mathrm{e}^{-t/2}\sin(2t)$$

(b) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$, with $y(0) = 1$ and $y'(0) = -1$.
SOLUTION: Take the Laplace transform of both sides:

$$(s^2Y - s + 1) + (sY - 1)\tfrac{5}{4}Y = \frac{1}{s} - \frac{\mathrm{e}^{-\pi s}}{s}$$

so that

$$Y(s) = \frac{1-\mathrm{e}^{-\pi s}}{s(s^2+s+5/4)} + \frac{s}{s^2+s+5/4}$$

For the second term,

$$\frac{s}{s^2+s+5/4} = \frac{s}{(s+\frac{1}{2})^2+1} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+1}$$

For the first term, treat it like:

$$\left(\mathrm{e}^{-0s} - \mathrm{e}^{-\pi s}\right)H(s)$$

where

$$H(s) = \frac{1}{s}\cdot\frac{1}{s^2+s+\frac{5}{4}} = \frac{1}{s}\cdot\frac{1}{(s+\frac{1}{2})^2+1}$$

so that

$$h(t) = 1 * \mathrm{e}^{-t/2}\sin(t)$$

Therefore, the overall answer is:

$$y(t) = h(t) - u_\pi(t)h(t-\pi) + \mathrm{e}^{-t/2}\left(\cos(t) - \frac{1}{2}\sin(t)\right)$$

11. Short Answer:

(a) $\int_0^\infty \sin(3t)\delta(t-\frac{\pi}{2})\,dt = \sin(3\pi/2) = -1$, since

$$\int_0^\infty f(t)\delta(t-c)\,dt = f(c)$$

(b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t-c), \qquad y(0) = 0$$

SOLUTION:

$$sY = \mathrm{e}^{-cs} \Rightarrow Y = \frac{\mathrm{e}^{-cs}}{s}$$

so that $y(t) = u_c(t)$. Therefore, the "derivative" of the Heaviside function is the Dirac $\delta-$function!

(c) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2+2x+5)$ if the series is based at $x_0 = 1$?

SOLUTION (From 5.4):

The roots of the denominator are where $x^2+2x+5=0$. Use the quadratic formula or complete the square to find the roots,

$$(x+1)^2 = -4 \Rightarrow x = -1 \pm 2i$$

Find the distance (in the complex plane) between $x = 1$ and either root (the distances will be the same). In this case,

$$\sqrt{(1--1)^2 + (0-2)^2} = \sqrt{8}$$

Therefore, the radius of convergence is $2\sqrt{2}$.

(d) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t-x) f(x)\,dx = \mathrm{e}^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2+1} = \frac{1}{s+1} \qquad \Rightarrow \qquad F(s)\left(\frac{(s+1)^2}{s^2+1}\right) = \frac{1}{s+1}$$

so that

$$F(s) = \frac{s^2+1}{(s+1)^3}$$

(e) In order for the Laplace transform of f to exist, f must be?

f must be piecewise continuous and of exponential order

(f) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

SOLUTION: No. Notice that the second derivative is not continuous at $x = 3$, but the second derivative of the power series would be.

(g) Use the table to find the Laplace transform of $\mathrm{e}^{-2t}\sinh(\sqrt{3}\,t)$.

SOLUTION: Use Table Entries 14 and 7:

$$\mathcal{L}\left(\mathrm{e}^{-2t}\sinh(\sqrt{3}t)\right) = F(s+2)$$

where $F(s)$ is the Laplace transform of $\sinh(\sqrt{3}t)$:

$$F(s) = \frac{\sqrt{3}}{s^2-9} \qquad \Rightarrow \qquad \text{Overall Answer:} \qquad F(s+2) = \frac{\sqrt{3}}{(s+2)^2-9}$$

(h) Is $x = 0$ an ordinary point of $xy'' + 3x^2y' + y = 4$?

SOLUTION: No. Dividing by x gives $p(x) = 3x$, but $q(x) = 1/x$, so we cannot have $x = 0$ as an ordinary point (so $x = 0$ is a singular point).

12. Let $f(t) = t$ and $g(t) = u_2(t)$.

(a) Use the Laplace transform to compute $f * g$.

To use the table,

$$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{\mathrm{e}^{-2s}}{s} = \mathrm{e}^{-2s}\frac{1}{s^3} = \mathrm{e}^{-2s}H(s)$$

so that $h(t) = \frac{1}{2}t^2$. The inverse transform is then

$$u_2(t)\frac{1}{2}(t-2)^2$$

(b) Verify your answer by directly computing the integral.

By direct computation, we'll choose to "flip and shift" the function t:

$$f * g = \int_0^t (t-x)u_2(x)\,dx$$

Notice that $u_2(x)$ is zero until $x = 2$, then $u_2(x) = 1$. Therefore, if $t \leq 2$, the integral is zero. If $t \geq 2$, then:

$$\int_0^t (t-x)u_2(x)\,dx = \int_2^t t - x\,dx = tx - \frac{1}{2}x^2\Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$

valid for $t \geq 2$, zero before that. This means that the convolution is:

$$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

13. If $a_0 = 1$, determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

SOLUTION: The recognition problem is a little difficult, but we should be able to get the coefficients:

$$a_{k+2} = -\frac{2}{k+1}a_k$$

Just doing the straight computations, we get:

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \cdots$$

To see the pattern, it is easiest to look at the general terms (Typically, I wouldn't ask for the recognition part on the exam, but you should be able to get the first few computations, as we did above):

$$\begin{array}{lll}
a_1 & = -2a_0 & = \frac{(-2)}{1!}a_0 \\
a_2 & = -\frac{2}{2}a_0 = 2a_0 & = \frac{4}{2!}a_0 \\
a_3 & -\frac{2}{3}a_2 = -\frac{4}{3}a_0 & = \frac{-8}{3!}a_0 \\
\vdots & & \vdots
\end{array}$$

The series is for $\mathrm{e}^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

14. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k(x-1)^k$ (with a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + x(x-2)\sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In front of the second sum we have $x^2 - 2x$, but we can't bring that directly into the sum since we have powers of $(x-1)$. But, we might recognize that:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x-1)^2 - 1$$

Therefore, the second sum can be expanded into two sums:

$$((x-1)^2 - 1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = (x-1)^2\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

Now we have three sums to work with

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In the first sum, the first non-zero term has $(x-1)^0$, the second sum begins with $(x-1)^2$, and the last sum starts with $(x-1)^0$. We could shift the second index to start at $n = 0$, but then the sum begins with $(x-1)^1$. We'll have to break off the constant terms from the first two sums:

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1)(x-1)^{n-2}$$

and similarly

$$-\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = -a_1 - \sum_{n=2}^{\infty} na_n(x-1)^{n-1}$$

Now we can bring all three sums together. In the first sum, we'll substitute $k = n-2$ (or $n = k+2$). In the middle sum, $k = n+1$ (or $n = k-1$), and in the third sum, $k = n-1$ (or $n = k+1$). With these substitutions, we get:

$$2a_2 - a_0 + \sum_{k=1}^{\infty} \left((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} - (k+1)a_{k+1}\right)(x-1)^k$$

NOTE: The question asked for the index to start at $k = 2$ instead of $k = 1$- It's OK to do it either way; mainly, I wanted to see you put the sums together as one.

15. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \qquad y(0) = 1, y'(0) = 1$$

SOLUTION: The idea behind this question is to get you to think about the kinds of solutions we get from the Laplace transform. If we do not require y to be continuous, then this DE is actually two differential equations:

$$y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 1 \qquad \text{valid for } t \leq 1$$

And

$$y'' + 4y = 1 \qquad y(1), y'(1) \text{ arbitrary }, \text{ valid for } t > 1$$

The general solution is then:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

If we require $y(t)$ to be continuous (a very common assumption), then we get the answer that comes from using Laplace transforms. Writing the answer in piecewise form:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ -\frac{1}{4}\cos(2(t-1)) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

16. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

SOLUTION: For the first part, use Table Entry 19. In particular,

$$\mathcal{L}(tf(t)) = -F'(s)$$

where, in our case, $f(t) = y'(t)$, so that $F(s) = sY - y(0)$. Therefore,

$$\mathcal{L}(tf(t)) = -(Y - sY') = sY' - Y$$

Substituting this into the DE, we get:

$$Y' + \left(\frac{s^2 - 3s - 6}{3s}\right) Y = \frac{1}{s}$$

17. Exercises with the table:

   (a) Use table entries 5 and 14 to prove the formula for 9.
   SOLUTION: Prove formula #9 using 5 and 14:

$$\mathcal{L}(\mathrm{e}^{at}\sin(bt)) = F(s-a)$$

   where

$$F(s) = \mathcal{L}(\sin(bt)) = \frac{b}{s^2 + b^2} \quad \Rightarrow \quad \frac{b}{(s-a)^2 + b^2}$$

   Therefore,

$$\mathcal{L}(\mathrm{e}^{at}\sin(bt)) = \frac{b}{(s-a)^2 + b^2}$$

(b) Show that you can use table entry 19 to find the Laplace transform of $t^2\delta(t-3)$ (verify your answer using a property of the δ function).

SOLUTION: Using Entry 19, the Laplace transform of $t^2\delta(t-3)$ is the second derivative of the Laplace transform of $\delta(t-3)$. That is, using

$$F(s) = \mathrm{e}^{-3s}$$

then

$$\mathcal{L}(t^2\delta(t-3)) = F''(s) = 9\mathrm{e}^{-3s}$$

And this is the same as:

$$\int_0^\infty \mathrm{e}^{-st}t^2\delta(t-3)\,dt = 9\mathrm{e}^{-3s}$$

(c) Prove (using the definition of $\mathcal{L}$) table entries 12 and 13.

SOLUTION: 12 is a special case of 13, so we prove 13 using the definition:

$$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty \mathrm{e}^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty \mathrm{e}^{-st}f(t-c)\,dt$$

We want this answer to be the following (with a different variable of integration):

$$\mathrm{e}^{-cs}F(s) = \mathrm{e}^{-cs}\int_0^\infty \mathrm{e}^{-sw}f(w)\,dw = \int_0^\infty \mathrm{e}^{-s(w+c)}f(w)\,dw$$

We can connect the two by taking $w = t-c$ (so that $t = w+c$), and then (remember to change the bounds!):

$$\int_c^\infty \mathrm{e}^{-st}f(t-c)\,dt = \int_0^\infty \mathrm{e}^{-s(w+c)}f(w)\,dw$$

And we're done.

(d) Prove (using the definition of $\mathcal{L}$) a formula (similar to 18) for $\mathcal{L}(y'''(t))$.

SOLUTION: I wanted you to recall how we got those definitions in the past (integrating by parts):

$$\mathcal{L}(y'''(t)) = \int_0^\infty \mathrm{e}^{-st}y'''(t)\,dt$$

Integration by parts using a table:

$$\begin{array}{crr} + & \mathrm{e}^{-st} & y'''(t) \\ - & -s\mathrm{e}^{-st} & y''(t) \\ + & s^2\mathrm{e}^{-st} & y'(t) \\ - & -s^3\mathrm{e}^{-st} & y(t) \end{array} \quad \Rightarrow \quad \left(\mathrm{e}^{-st}\left(y''(t) + sy'(t) + s^2y(t)\right)\right|_{t=0}^{\infty} + s^3\int_0^\infty \mathrm{e}^{-st}y(t)\,dt$$

At infinity, these terms all go to zero (otherwise, the Laplace transform wouldn't exist), so we get:

$$s^3 - (y''(0) + sy'(0) + s^2y(0)) = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

18. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

Substituting our power series in for y, y', y'':

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Noting that in the second sum we could start at $n = 0$, since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} \left(2(k+2)(k+1)a_{k+2} + ka_k + 3a_k\right) x^k = 0$$

From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}\, a_k$$

(b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$

Substituting our power series in for y, y', y'':

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

The two middle sums can have their respective index taken down by one (so that formally the series would start with $0x^0$):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now make all the indices the same. To do this, in the first sum make $k = n-2$, in the second sum take $k = n-1$. Doing this and collecting terms:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + (k-1)a_k\right) x^k = 0$$

So we get the recursion:

$$a_{k+2} = \frac{(k+1)k\, a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c) $y'' - xy' - y = 0$, $x_0 = 1$

Done in class;

$$a_{n+2} = \frac{1}{n+2}\left(a_{n+1} + a_n\right)$$

19. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

Compute the derivatives directly, then (don't forget to divide by $n!$):

$$y(x) = 1 - x - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \ldots$$

20. Determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation:

$$(x^2 - 2x + 5)y'' + 4xy' + y = 0 \qquad x_0 = 0, \qquad x_0 = 3$$

SOLUTION: This was from 5.3. We see that the polynomial $x^2 - 2x + 5$ will be in the denominator, and it is zero where:

$$x^2 - 2x + 1 + 4 = 0 \quad \Rightarrow \quad (x-1)^2 = -4 \quad \Rightarrow \quad x = 1 \pm 2i$$

The radius is therefore the smaller of the distances between x_0 and the two roots (which will be the same):

If $x_0 = 0$, the distance is between $(0, 0)$ and $(1, 2)$: $\sqrt{1^2 + 2^2} = \sqrt{5}$

If $x_0 = 3$, the distance is between $(3, 0)$ and $(1, 2)$: $\sqrt{2^2 + 2^2} = 2\sqrt{2}$

21. Find the radius of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} x^n$

SOLUTION:

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}}|x| = |x|$$

So by the ratio test, the series will converge (absolutely) if $|x| < 1$ (so the radius is 1).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

SOLUTION: Simplifying the limit in the ratio test, we get

$$\lim_{n\to\infty} 2\sqrt{\frac{n}{n+1}}|x+3| = 2|x+3|$$

Therefore, by the ratio test, the series will converge absolutely if $2|x+3| < 1$, or if $|x+3| < 1/2$ (and this is our radius). For the interval of convergence, we have to check the points $x = -7/2$ and $x = -5/2$ separately. For $x = -7/2$, the series diverges ($p-$test), and for $x = -5/2$, the series converges by the alternating series test.

NOTE: If you don't recall those tests, you probably ought to review them, but I won't make you recall them for the exam this week.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{n^n}$ (A little tricky)

SOLUTION: This one is a little tricky because we need to recall the definition of e:

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

In this case, using the Ratio Test, we have:

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}\,|x| = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)(n+1)^n}\,|x| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x| = \frac{|x|}{e}$$

so the radius of convergence is e.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$

SOLUTION: This is definitely similar to problems on exams/quizzes. The Ratio Test simplifies to:

$$\frac{1}{5} \lim_{n\to\infty} \frac{n}{n+1}\,|3x-2| = \frac{|3x-2|}{5}$$

To converge absolutely, $|3x - 2| < 5$, so the radius of convergence is 5. Now we have to check the endpoints separately, which are $x = -1$ and $x = 7/3$:

- At $x = -1$, the sum becomes:

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

  This is an alternating harmonic series, which converges (but not absolutely).
- At $x = 7/3$, the sum becomes a harmonic series, which diverges.

The interval of convergence is: $[-1, \frac{7}{3})$

22. For the exercises from 5.4, see the solution set online.