Final Exam Review: Math 244

Instructions at the top of the exam will be the following:

> **Instructions:**
> A table of Laplace transforms and the Poicaré Diagram will be provided on a separate sheet. The Variation of Parameters formula will be provided on the board (as during the previous exam). You may use a calculator (with the restrictions previously discussed and certified to below).
>
> I certify that I did not use any computational device for symbolic manipulation or data storage:
>
> ________________________ ________________________
>
> Name Signature
>
> **Show all your work!**

Often, when we use the textbook to study, we may be unconsciously seeing the context of each question. Such hints will not be available during an exam, so the questions below are meant to give you examples of problems out of context, so you can practice putting them into the correct context to solve.

These are not meant to be exhaustive, but they are probably pretty similar to the kinds of questions that will be on the exam.

You should go through the three exam study guides and question sets, the old exams, and the old quizzes. If you get stuck on anything below, you should try to go back to the book for more types of questions like that.

Students often ask about how the material will be weighted- There is some crossover between the latest material and Chapters 2 and 3, but generally speaking the exam will be weighted about 30% on the latest material, about 30% on Chapters 5 and 6, and about 40% on Chapters 2 and 3.

1. Solve (use any method if not otherwise specified):
   (a) $-t\cos(t)\,dt + (2x - 3x^2)\,dx = 0$
   (b) $y'' + 2y' + y = \sin(3t)$
   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   (d) $y' = \sqrt{t}\mathrm{e}^{-t} - y$
   (e) $x' = 2 + 2t^2 + x + t^2 x$
2. Show that with the proper substitution, the following equation becomes separable. NOTE: You do not need to solve the differential equation.
$$\frac{dy}{dx} = \frac{3x - 4y}{y - 2x}$$
3. Show that with the proper substitution, the following equation becomes linear. NOTE: You do not need to solve the differential equation.
$$\frac{dy}{dx} + 3xy = \frac{x}{y^2}$$
4. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:
$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$
5. If $y' = y(1-y)(2-y)(3-y)(4-y)$ and $y(0) = 2.5$, determine what y does as $t \to \infty$.
6. If y_1, y_2 are a fundamental set of solutions to
$$t^2 y'' - 2y' + (3 + t)y = 0$$
and if $W(y_1, y_2) = 3$, find $W(y_1, y_2)(4)$.
7. Let $y''' - y' = t\mathrm{e}^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, y_p using Undetermined Coefficients. Do not solve for the coeffs.
8. What is the Wronskian? How is it used?
9. Explain Abel's Theorem:
10. Give the three Existence and Uniqueness Theorems we have had in class.
11. Let $y'' - 6y' + 9y = F(t)$. For each $F(t)$ listed, give the *form* of the general solution using undet. coeffs (do not solve for the coefficients).

(a) $F(t) = 2t^2$
(b) $F(t) = te^{-3t}\sin(2t)$
(c) $F(t) = t\sin(2t) + \cos(2t)$
(d) $F(t) = 2t^2 + 12e^{3t}$

12. Consider the predator-prey system:

$$\begin{aligned} x' &= x(1 - 0.5x - 0.5y) \\ y' &= y(-0.25 + 0.5y) \end{aligned}$$

(a) Which is predator, which is prey? How are the populations being modeled (what assumptions)?
(b) Find the equilibrium solutions.
(c) Classify the equilibria using the Poincaré diagram.
(d) What will happen to the population $x(t)$ for almost every starting point (as $t \to \infty$)?

13. Consider the system:

$$\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$$

See if you can sketch the direction field (find/classify the equilibria first!).

14. A spring is stretched 0.1 m by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 0.05 m below its resting equilibrium and released with a downward velocity of 0.1 m/s, determine its position u at time t.

15. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0$, $x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

16. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0$, $x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the Taylor series for the solution by computing derivatives.

17. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

18. Determine the Laplace transform:

(a) $t^2 e^{-9t}$
(b) $u_5(t)(t-2)^2$
(c) $e^{3t}\sin(4t)$
(d) $e^t \delta(t-3)$

19. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2 - 4s + 6}$
(b) $\dfrac{7}{(s+3)^3}$
(c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$
(d) $\dfrac{3s-2}{(s-4)^2 - 3}$

20. Solve the given initial value problems using Laplace transforms:

(a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$
(b) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
(c) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$ (You may write the solution as a convolution)

21. Consider

$$t^2 y'' - 4ty' + 6y = 0$$

(a) Thinking of this as an Euler equation, find the solution.
(b) Using $y_1 = t^2$ as one solution, find y_2 by computing the Wronskian two ways.
(c) Using $y_1 = t^2$ as one solution, find y_2 by using Variation of Parameters.

22. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$
(b) $y' = \frac{1}{2}y(3 - y)$
(c) $y'' + 2y' + y = 0, \quad y(0) = \alpha, y'(0) = 1$
(d) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$
(e) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1.$

23. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.

24. True or False (and explain): Every separable equation is also exact. If true, is one way easier to solve over the other?

25. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

26. A sky diver weighs 180 lbs and falls vertically downward. Assume that the constant for air resistance is 3/4 before the parachute is released, and 12 after it is released at 10 sec. Assume velocity is measured in feet per second, and $g = 32$ ft/sec^2.

    (a) Find the velocity of the sky diver at time t (before the parachute opens).

    (b) If the sky diver fell from an altitude of 5000 feet, find the sky diver's position at the instant the parachute is released.

    (c) After the parachute opens, is there a limiting velocity? If so, find it. (HINT: You do not need to re-solve the DE).

27. Rewrite the following differential equations as an equivalent system of first order equations. If it is an IVP, also determine initial conditions for the system.

    (a) $y'' - 3y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

    (b) $y''' - 2y'' - y' + 4y = 0$

    (c) $y'' - yy' + t^2 = 0$

28. Convert one of the variables in the following systems to an equivalent higher order differential equation, and solve it (be sure to solve for both x and y):

$$\begin{aligned} x' &= 4x + y \\ y' &= -2x + y \end{aligned}$$

29. Solve the previous system by using eigenvalues and eigenvectors.

30. Verify by direct substitution that the given power series is a solution of the differential equation:

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$(x+1)y'' + y' = 0$$

31. Convert the given expression into a single power series:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=2}^{\infty} n a_n x^{n-2} + 3\sum_{n=1}^{\infty} a_n x^n$$

32. Find the recurrence relation for the coefficients of the power series solution to $y'' - (1+x)y = 0$ at $x_0 = 0$.

33. Find the first 5 non-zero terms of the series solution to $y'' - (1+x)y = 0$ if $y(0) = 1$ and $y'(0) = -1$ (use derivatives).

34. We have two tanks, A and B with 20 and 30 gallons of fluid, respectively. Water is being pumped into Tank A at a rate of 2 gallons per minute, 2 ounces of salt per gallon. The well-mixed solution is pumped out of Tank A and into Tank B at a rate of 4 gallons per minute. Solution from Tank B is entering Tank A at a rate of 2 gallons per minute. Water is being pumped into Tank B at k gallons per minute with 3 ounces of salt per gallon. The solution is being pumped out of tank B at a total rate of 5 gallons per minute (2 of them are going into tank A).

    - What should k be in order for the amount of solution in Tank B to remain at 30? Use this value for the remaining problems.
    - Write the system of differential equations for the amount of salt in Tanks A, B at time t. Do not solve.
    - Find the equilibrium solution and classify it.

35. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

36. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, if A is as defined below:

    (a) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

    (c) $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$

37. For each nonlinear system, find and classify the equilibria:

    (a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1+2y \\ 1-3x^2 \end{bmatrix}$

    (b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-y \\ x^2-y^2 \end{bmatrix}$

    (c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+x^2+y^2 \\ y(1-x) \end{bmatrix}$

    (d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-y^2 \\ y-x^2 \end{bmatrix}$