Solutions to the Review Questions

Short Answer/True or False

1. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.
   False: (a) Equilibrium solutions are only defined for *autonomous* differential equations, (b) This is an isocline associated with a slope of zero, and (c) $y = -2t$ is not a solution.

   (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t. (If true, say why. If False, solve the DE).
   False. The E & U Theorem tells that a unique solution will exist for any initial condition (since $1 + y^2$ and $2y$ are continuous everywhere), but it does not say on what interval the solution will exist. For example, if we take $y(0) = y_0$ and solve, we get:

   $$\int \frac{dy}{1+y^2} = \int dt \quad \Rightarrow \quad \tan^{-1}(y) = t + C \quad \Rightarrow \quad C = \tan^{-1}(y_0)$$

   Therefore,

   $$y = \tan(t + \tan^{-1}(y_0))$$

   where

   $$-\frac{\pi}{2} < t + \tan^{-1}(y_0) < \frac{\pi}{2}$$

   (so that the tangent function is invertible).

   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume df/dy is continuous).
   False. Suppose there is a stable equilibrium at $y = a$ and at $y = b$. At $y = a$, $f(y)$ must cross the $y-$axis from a positive value to a negative (draw a sketch), and similarly, at $y = b$, f must cross from positive to negative. Therefore, f is going from a slightly negative number to the right of $y = a$ to a slightly positive number to the left of $y = b$. Since we're assuming f is continuous, the Intermediate Value Theorem states that f must cross the $y-$axis between a and b, and in fact, this crossing must be from a negative to a positive number (which is an unstable equilibrium).

   (d) All autonomous equations are separable.
   True. Any autonomous equation can be written as $y' = f(y) \cdot 1$, which is separable and $\displaystyle\int \frac{dy}{f(y)} = \int dt$.

   (e) All linear (first order) equations are Bernoulli.
   True- It is possible to categorize them this way:

   $$y' + p(t)y = g(t)y^0$$

   but we typically will not write them this way (so you could argue "false" as well).

   (f) All separable equations are exact.
   True. If the equation is separable, then $y' = f(y)g(x)$, which can be written:

   $$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \quad \Rightarrow \quad -g(x) + \frac{1}{f(y)}\frac{dy}{dx} = 0$$

   Now, if $M(x, y) = -g(x)$, then $M_y = 0$, and $N(x, y) = 1/f(y)$ means $N_x = 0$.

2. The Existence and Uniqueness Theorems:

   - Linear: $y' + p(t)y = g(t)$ at (t_0, y_0):
     If p, g are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem and that solution is valid for all t in the interval I.

- General Case: $y' = f(t, y)$, (t_0, y_0):

  Let the functions f and f_y be continuous in some open rectangle R containing the point (t_0, y_0). Then there exists an interval about t_0, $(t_0 - h, t_0 + h)$ contained in R for which a unique solution to the IVP exists.

  Side Remark 1: To determine such a time interval, we must solve the DE.

  Side Remark 2: We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

3. To solve $y' = y^{1/3}$, we separate variables:

$$y^{-1/3}\, dy = dt$$

Before going further, it is good practice to note that the previous step is valid, *as long as* $y \neq 0$. The case that $y = 0$ can be taken separately- In fact, we see that $y(t) = 0$ is an equilibrium solution that satisfies the initial condition.

Going on, we integrate:

$$\frac{3}{2}y^{2/3} = t + C_1 \quad \Rightarrow \quad y^{2/3} = \frac{2}{3}t + C_2 \quad \Rightarrow \quad y = \left(\frac{2}{3}t + C_2\right)^{3/2}$$

We can solve for C_2 using the initial condition: $0 = C_2$, so that

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

We can verify that this is indeed a solution by substituting it back into the DE (not necessary; just a way of double-checking yourself):

$$y' = \frac{3}{2}\left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{1/2}$$

And on the other hand,

$$y^{1/3} = \left[\left(\frac{2t}{3}\right)^{3/2}\right]^{1/3} = \left(\frac{2t}{3}\right)^{1/2}$$

Therefore, this is indeed a second solution to the IVP.

Of course, the Existence and Uniqueness Theorem cannot be "violated" since it is a theorem, but in this case, the *hypotheses* are not satisfied:

$$y' = f(t, y) \quad \Rightarrow \quad f(t, y) = y^{1/3}$$

In this case, f is continuous at $(0, 0)$ but $\partial f/\partial y$ is not.

Solve:

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

   Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x + y)\, dx - (x - y)\, dy = 0$. Hint: Let $v = y/x$.

   Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1 + v}{1 - v}$$

With the substitution $xv = y$, we get the substitution for dy/dx:

$$v + xv' = y'$$

So that the DE becomes:

$$v + xv' = \frac{1+v}{1-v} \quad \Rightarrow \quad xv' = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1+v} = \frac{1+v^2}{1-v}$$

The equation is now separable:

$$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln|x| + C$$

Therefore,

$$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

Lastly, back-substitute $v = y/x$.

3. $\dfrac{dy}{dx} = \dfrac{2x+y}{3+3y^2-x} \qquad y(0) = 0.$

   This is exact. The solution is, with $y(0) = 0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\dfrac{dy}{dx} = -\dfrac{2xy+y^2+1}{x^2+2xy}$

   This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0) = 2$

   This is linear and separable. $y(t) = \frac{2}{3}\sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. State the interval on which the solution is valid.

   This is linear and separable. As a linear equation, the solution will be valid on all t (since $p(t) = -\frac{1}{2}$). The solution is $y(t) = 200\mathrm{e}^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2}\,dy = \int (1-2x)\,dx \quad \Rightarrow \quad -\frac{1}{y} = x - x^2 + C$$

   Put in the initial condition (IC): $6 = 0 + C$. Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

   The solution is valid on the interval $(-2, 3)$.

8. $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$

   This is linear (but not separable). $y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{(1/2)t}$

9. $y' = \frac{1}{2}y(3-y)$

   Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int dt$$

   Simplify your answer for y by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)\mathrm{e}^{-(3/2)t} + 1}$$

10. $\sin(2t)\ dt + \cos(3y)\ dy = 0$

Separable (and/or exact): $\frac{-1}{2}\cos(2t) + \frac{1}{3}\sin(3y) = C$

11. $y' = xy^2$

Separable: $y = \dfrac{1}{-(1/2)x^2 - C}$

12. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$ (Hint: Think Bernoulli)

SOLUTION: Multiply by y to get

$$yy' - \frac{3}{2x}y^2 = 2x$$

So (following what we did for the general Bernoulli eqn), let $v = y^2$, and therefore $v' = 2yy'$. Multiply by 2 to get the right form, then substitute

$$2yy' - \frac{3}{x}y^2 = 4x \quad \Rightarrow \quad v' - \frac{3}{x}v = 4x$$

Now it is a standard linear DE. Solving, we get $v = -4x^3 + Cx^3$, and

$$y^2 = -4x^3 + Cx^3$$

(We'll leave in implicit form).

13. $yy'' = (y')^2$ With the hint, we need to find a substitution for y'' (and then leave the equation as a DE with unknown function p with variable y):

$$\frac{dy}{dt} = p(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{dp}{dy}\cdot\frac{dy}{dt} = \frac{dp}{dy}p(y)$$

Now, substitute in the expressions and divide by $yp(y)$:

$$y\frac{dp}{dy}p(y) = (p(y))^2 \quad \Rightarrow \quad \frac{dp}{dy} = \frac{1}{y}p(y) \quad \Rightarrow \quad \int \frac{1}{p}\,dp = \int \frac{1}{y}\,dy \quad \Rightarrow \quad \ln|p| = \ln|y| + C \quad \Rightarrow \quad p = Ay$$

where A is a constant of integration. Now, substitute again with $p = y'$, and we get:

$$y' = Ay \quad \Rightarrow \quad y = P\mathrm{e}^{At}$$

where P is another constant of integration.

We can verify our answer as well: $y' = AP\mathrm{e}^{At}$, and $y'' = A^2P\mathrm{e}^{At}$ so that

$$yy'' = P\mathrm{e}^{At}\cdot A^2P\mathrm{e}^{At} = A^2P^2\mathrm{e}^{2At}$$

and this expression is also $(y')^2$.

14. $y' + 2y = g(t)$ with $y(0) = 0$ and $g(t) = 1$ on $0 \le t \le 1$ and zero elsewhere.

SOLUTION: This is similar to Exercise 33, Section 2.4. In this case, we go ahead and solve starting at time 0:

$$y' + 2y = 1 \quad \Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t}$$

and this is valid for $0 \le t \le 1$. When we hit $t = 1$, the dynamics change to:

$$y' + 2y = 0 \quad \Rightarrow \quad y(t) = P\mathrm{e}^{-2t}$$

Now we will typically choose the constants so that y is continuous. Therefore, using our previous function, $y(1) = (1 - \mathrm{e}^{-2})/2$, and our current function: $y(1) = P\mathrm{e}^{-2}$, or

$$P = \frac{e^2 - 1}{2}$$

Therefore, the overall solution to the DE would be:

$$y(t) = \begin{cases} (1 - e^{-2t})/2 & \text{if } 0 \le t \le 1 \\ ((e^2 - 1)/2)\mathrm{e}^{-2t} & \text{if } 1 > 1 \end{cases}$$

Just for fun, the direction field and solution curve are plotted in Figure 1.

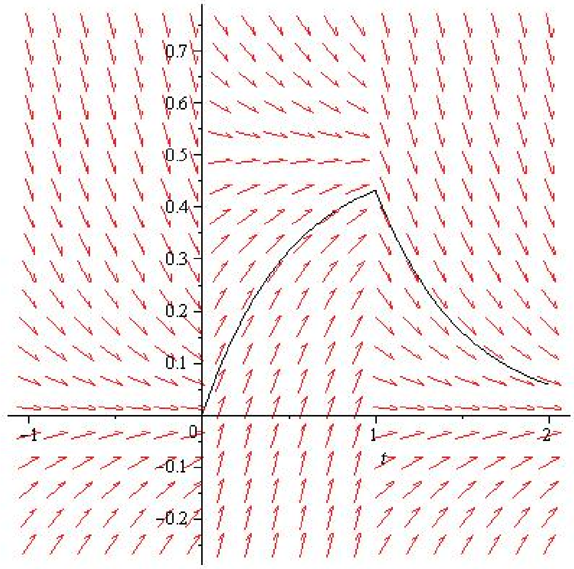

Figure 1: Direction field and solution curve for Exercise 14. Note how the solution approaches one equilibrium until $g(t)$ changes, then it goes to the new equilibrium.

Misc.

1. Construct a linear first order differential equation whose general solution is given by:

   (a) $y(t) = t - 3 + \frac{C}{t^2}$

   SOLUTION: Construct y'. The idea will be to produce a linear DE. Therefore, we need to construct y' and compare it to y:

   $$y' = 1 - 2Ct^{-3}$$

   Add this to some multiple (t's are allowed) of y to get of the arbitrary constant. In this case,

   $$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

   or,

   $$ty' + 2y = 3t - 6$$

   (b) $y(t) = 2\sin(3t) + C\mathrm{e}^{-2t}$

   SOLUTION: Same idea as before. Try to get y and y' together in such a way as to cancel out the arbitrary C:

   $$y' = 6\cos(3t) - 2C\mathrm{e}^{-2t}$$

   so that: $y' + 2y = 4\sin(3t) + 6\cos(3t)$.

2. Construct a linear first order DE so that all solutions tend to $y = 3$ as $t \to \infty$.

   SOLUTION: One way to think about it is to construct $y' = ay + b$, which is also autonomous- In fact, we are saying that we need a line in the plane, through $y = 3$ so that the equilibrium is stable- That is any line through $(0, 3)$ with a negative slope. For example, $y' = -y + 3$

3. Suppose we want to construct a population model so that there is a logistic population growth-

   (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.

   SOLUTION: In the (y, y') plane, we are looking at an upside down parabola that goes through $y = 0$ and $y = 100$. One model is therefore

   $$y' = y(100 - y)$$

(b) Using your model, now assume there is a continuous"harvest" (of k units per time period). How does that effect the model- In particular, is there a critical value of k over which the population will be extinct? If so, find it.

SOLUTION: This subtracts k from our population:

$$y' = y(100 - y) - k$$

We see that for some value of k, the vertex of the upside parabola is exactly on the $y-$axis. We can solve this by completing the square- That is, we should be able to write y' as a perfect square:

$$-y^2 + 100y + k \quad \rightarrow \quad -(y^2 - 100y + 50^2) = -(y - 50)^2$$

Therefore, $k = 50^2 = 2500$. This is the maximum allowable harvest before the ecosystem collapses.

4. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

   Write the initial value problem that describes the amount of salt in the tank at time t, and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \qquad Q(0) = Q_0$$

   The solution is:

$$Q = rM + (Q_0 - rM)\mathrm{e}^{-(\gamma/M)t}$$

5. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0? Does this make sense?

   Note that the differential equation for Q is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as $t \to \infty$ and see that $Q \to rM$. This does not necessarily depend on Q_0; if Q_0 starts at equilibrium, rM, then Q is constant.

   It does make sense. The incoming concentration of salt is r pounds per gallon, so we would expect the long term concentration to be the same, $rM/M = r$.

6. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

$$\frac{dQ}{dt} = 4 - \frac{3}{100 - t}Q \qquad Q(0) = 50$$

   This goes from being autonomous to linear. In this case, use an integrating factor,

$$\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{3\int \frac{1}{100-t}\,dt} = \mathrm{e}^{-3\ln|100-t|} = (100 + t)^{-3} \qquad t > 100$$

   Going back to the DE

$$\left(\frac{Q}{(100 - t)^3}\right) = 4(100 - t)^{-3} \quad \Rightarrow \quad Q = 2(100 - t) + C(100 - t)^3$$

   Continuing, we get:

$$Q(t) = 2(100 - t) - \frac{150}{100^3}(100 - t)^3$$

7. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t.

   The general model is: $mv' = mg - kv$. In this case, $m = 1$, $g = 9.8$ and $k = 1/2$. Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

   Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

   Solve it:

$$v(t) = 19.6\left(1 - \mathrm{e}^{-(1/2)t}\right)$$

8. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

   SOLUTION: $S' = rS - k$, where S will be the amount owing, r is the annual interest rate and k is the rate for the continuous payment (in years). Then using $S(0) = S_0$, we can write the solution as

$$S(t) = \frac{k}{r} + \left(S_0 - \frac{k}{r}\right) e^{rt}$$

   Substituting in the values for r and S_0, and $t = 10$, we can solve the equation for k (note: $k/r = 20k$):

$$0 = 20k + (10000 - 20k)e^{-0.05 \cdot 10}$$

   Extra: If we go ahead and solve, we get a monthly payment rate of approximately \$105.90.

9. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.

   (a) Find and classify the equilibrium.
   SOLUTION: From the sketch given, $y = 0$ is asymptotically stable and $y = 1$ is semistable.

   (b) Find intervals (in y) on which $y(t)$ is concave up.
   SOLUTION: Examine the intervals $y < 0$, $0 < y < 1/3$, $1/3 < y < 1$ and $y > 1$ separately. The function y will be concave up when dF/dy and F both have the same sign- This happens when F is either increasing and positive (which happens nowhere) or decreasing and negative:

$$0 < y < \frac{1}{3} \qquad y > 1$$

   (c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. See the figure below.

   (d) Find an appropriate polynomial for $F(y)$.
   SOLUTION: One example is

$$y' = -y(y-1)^2$$

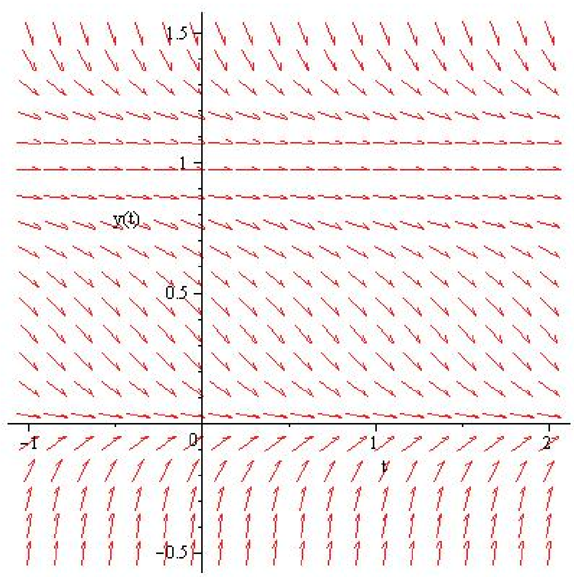

Figure 2: Figure for Exercise 9.

10. Consider the DE: $y\,dx + (2x - ye^y)\,dy = 0$. Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of y alone (Hint: Find the integrating factor first).

    SOLUTION: Let μ be the integrating factor, and assume μ is a function of y alone. Then

    $$(\mu M)_y = \mu' \cdot M + \mu \cdot M_y = y\mu' + \mu$$

    And

    $$(\mu N)_x = 0 \cdot N + \mu \cdot 2$$

    Setting these equal, we have the DE:

    $$y\mu' = \mu \quad \Rightarrow \quad \int \frac{1}{\mu}\,d\mu = \int \frac{dy}{y}$$

    or $\mu = y$. We can verify our answer, since $y^2\,dx + (2xy - y^2e^y)\,dy = 0$ should now be exact.

11. Derive Euler's Method in stepping one unit in time. That is, given $y' = f(t, y)$, with $y(t_0) = y_0$, explain how we get $y_1 = y(t_1)$. Be sure to give an explanation, and not just write the formula down.

    SOLUTION: We use the tangent line approximation to $y(t)$ at t_0:

    $$y - y_0 = y'(t_0)(t - t_0) \quad \Rightarrow \quad y = y_0 + f(t_0, y_0)(t - t_0)$$

    then y_1 is approximated by substituting $t = t_1$:

    $$y_1 \approx y_0 + f(t_0, y_0)(t_1 - t_0)$$

12. Use Euler's Method to determine $y(1/2)$ if $y' = 4 - t + 2y$, with $y(0) = 1$ and if we use one step in time. Repeat if we use 2 steps in time.

    SOLUTION:

    $$y'(0) = f(0, 1) = 4 - 0 + 2 = 6$$

    so advancing 1/2 unit in time, we have:

    $$y(1/2) \approx 1 + 6 \cdot (1/2) = 1 + 3 = 4$$

    Taking two steps to do it, each step size would be 1/4:

    $$y(1/4) \approx 1 + 6 \cdot 1/4 = 5/2$$

    Now, $t = 1/4$ and $y = 5/2$, so the new slope is $f(1/4, 5/2) = 35/4$, and

    $$y(1/2) \approx \frac{5}{2} + \frac{35}{4} \cdot \frac{1}{4} \approx 4.7$$

13. Suppose we have a vector field

    $$\vec{F} = \langle 2x^2y + 2x, -(2xy^2 + 2y) \rangle$$

    and we drop a particle into the (x, y) plane at the point $(1, 2)$. Find a curve along which the particle will travel by first constructing an appropriate differential equation and showing it is exact.

    SOLUTION: First, the curve parametrized by $\langle x(t), y(t) \rangle$ will satisfy the DE:

    $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -(2xy^2 + 2y)/(2x^2 + 2x) \Rightarrow (2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

    which is exact, and the general solution is

    $$x^2y^2 + 2xy = k$$

    Using the initial condition $y(1) = 2$, we get $k = 8$, or the solution is

    $$x^2y^2 + 2xy = 8$$

14. Given the direction field below, find a differential equation that is consistent with it.

    SOLUTION: Draw the corresponding figure in the (y, y') plane first. There we see that $y = 0$ is unstable (make it a linear crossing), and $y = 2$ is stable, $y = 4$ is unstable. From the figure,

$$y' = y(y-2)(y-4)$$

    will work.

15. Consider the direction field below, and answer the following questions:

    (a) Is the DE possibly of the form $y' = f(t)$?
    SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at $t = -3$; the slopes are clearly not equal).

    (b) Is the DE possible of the form $y' = f(y)$?
    SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at $y = 1$- Some slopes are zero, others are not).

    (c) Is there an equilibrium solution? (If so, state it):
    SOLUTION: Yes- At $y = 0$.

    (d) Draw the solution corresponding to $y(-1) = 1$.
    SOLUTION: Just draw a curve consistent with the arrows shown.

16. Evaluate the following integrals:

    SOLUTIONS: You should be able to integrate by parts and use partial fractions fairly quickly at this point. Try checking your answers on Wolfram Alpha (for example, `integrate x^3exp(2x)`)

    Some notes: The first integral should be done using a table, and the last should simplify a lot.

$$\int x^3 e^{2x}\,dx \qquad \int \frac{x}{(x-1)(2-x)}\,dx \qquad e^{-3\int dt/t}$$