# Review Questions, Exam 1, Math 244

These questions are presented to give you an idea of the variety and style of question that will be on the exam. It is not meant to be exhaustive, so be sure that you understand the homework problems and quizzes.

## Discussion Questions:

1. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all $t$. (If true, say why. If False, solve the DE).

   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume $df/dy$ is continuous).

   (d) All autonomous equations are separable.

   (e) All linear (first order) equations are Bernoulli.

   (f) All separable equations are exact.

2. State the Existence and Uniqueness Theorem (for both linear and nonlinear IVPs)

3. Let $y' = y^{1/3}$, $y(0) = 0$. Find two solutions to the IVP. Does this violate the Existence and Uniqueness Theorem?

## Solve:

Give the general solution if there is no initial value. Before giving the solution, state what kind of differential equation it is (linear, separable, exact)- Multiple classes are possible, just give the one you will use.

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

2. $(x + y)\,dx - (x - y)\,dy = 0$. Hint: Homogeneous

3. $\dfrac{dy}{dx} = \dfrac{2x + y}{3 + 3y^2 - x} \qquad y(0) = 0.$

4. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0) = 2$

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. Additionally, give the interval on which the solution is valid.

7. $y' = (1 - 2x)y^2 \qquad y(0) = -1/6$. Additionally, give the interval on which the solution is valid.

8. $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$

9. $y' = \frac{1}{2}y(3-y)$

10. $\sin(2t)\ dt + \cos(3y)\ dy = 0$

11. $y' = xy^2$

12. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$ (Hint: Think Bernoulli)

13. $yy'' = (y')^2$ (Hint: Let $p(y) = y'$, and see if you can get the equation in terms of $p$ and $y$).

14. $y' + 2y = g(t)$, $y(0) = 0$ and

$$g(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

## Misc.

1. Construct a linear first order differential equation whose general solution is given by:
   (a) $y(t) = t - 3 + \frac{C}{t^2}$
   (b) $y(t) = 2\sin(3t) + Ce^{-2t}$

2. Construct a linear first order differential equation so that all solutions tend to $y = 3$ as $t \to \infty$.

3. Suppose we want to construct a population model so that there is a logistic population growth-
   (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.
   (b) Using your model, now assume there is a continuous"harvest" (of $k$ units per time period). How does that effect the model- In particular, is there a critical value of $k$ over which the population will be extinct? If so, find it.

4. Suppose we have a tank that contains M gallons of water, in which there is $Q_0$ pounds of salt. Liquid is pouring into the tank at a concentration of $r$ pounds per gallon, and at a rate of $\gamma$ gallons per minute. The well mixed solution leaves the tank at a rate of $\gamma$ gallons per minute.

   Write the initial value problem that describes the amount of salt in the tank at time $t$, and solve:

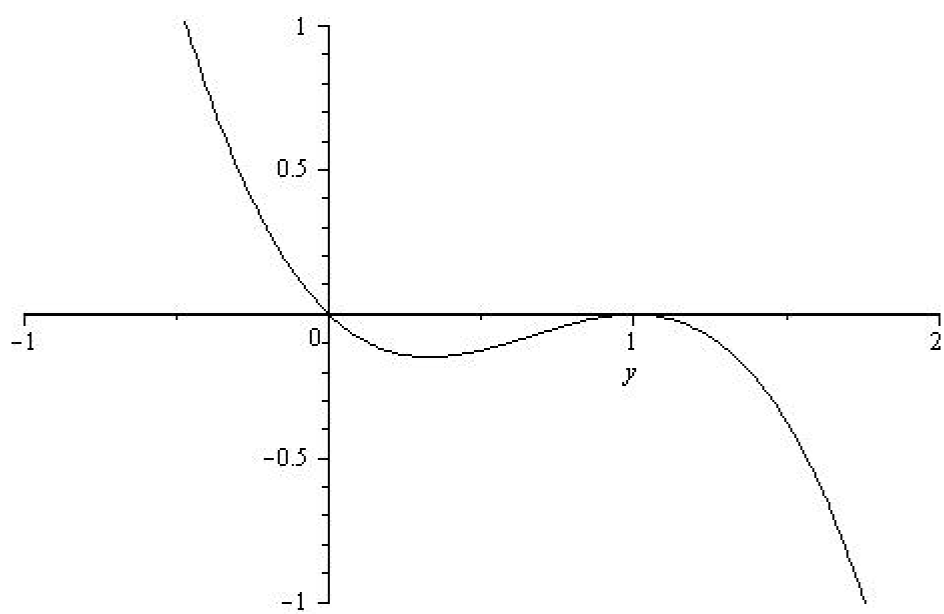
5. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on $Q_0$? Does this make sense?

6. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

7. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time $t$. In the $(t, y)$ plane, draw several solution curves (Hint: This is an autonomous DE).

8. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of $k$ dollars per month, set up a model for how much you owe at time $t$ in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

9. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.
   (a) Find and classify the equilibrium.
   (b) Find intervals (in $y$) on which $y(t)$ is concave up.
   (c) Draw a sketch of $y$ on the direction field, paying particular attention to where $y$ is increasing/decreasing and concave up/down.
   (d) Find an appropriate polynomial for $F(y)$.




10. Consider the DE: $y\,dx + (2x - ye^y)\,dy = 0$. Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of $y$ alone (Hint: Find the integrating factor first).

11. Derive Euler's Method in stepping one unit in time. That is, given $y' = f(t, y)$, with $y(t_0) = y_0$, explain how we get $y_1 = y(t_1)$. Be sure to give an explanation, and not just write the formula down.

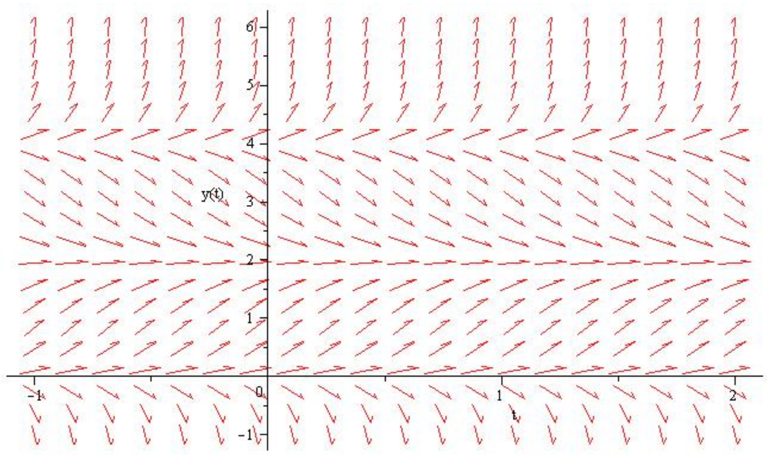
12. Use Euler's Method to determine $y(1/2)$ if $y' = 4 - t + 2y$, with $y(0) = 1$ and if we use one step in time. Repeat if we use 2 steps in time.

13. Suppose we have a vector field

$$\vec{F} = \langle 2x^2y + 2x, -(2xy^2 + 2y) \rangle$$

and we drop a particle into the $(x, y)$ plane at the point $(1, 2)$. Find a curve along which the particle will travel by first constructing an appropriate differential equation and showing it is exact.

14. Given the direction field below, find a differential equation that is consistent with it.



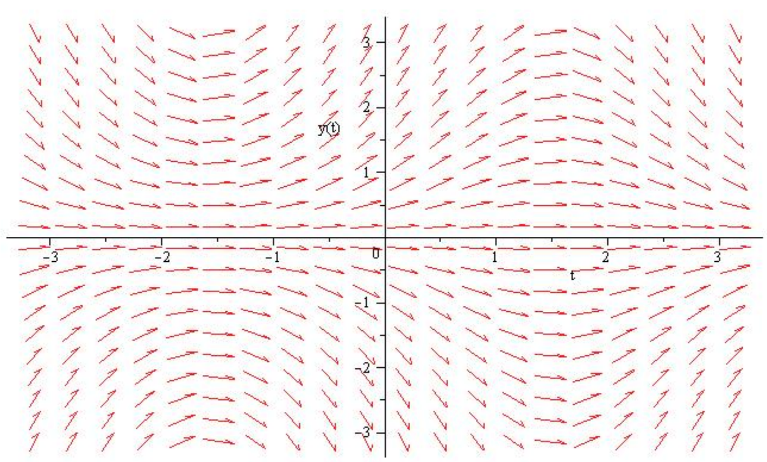

15. Consider the direction field below, and answer the following questions:

   (a) Is the DE possibly of the form $y' = f(t)$?

   (b) Is the DE possible of the form $y' = f(y)$?

   (c) Is there an equilibrium solution? (If so, state it):

   (d) Draw the solution corresponding to $y(-1) = 1$.




16. Evaluate the following integrals:

$$\int x^3 e^{2x}\, dx \qquad \int \frac{x}{(x-1)(2-x)}\, dx \qquad e^{-3\int dt/t}$$