

Solutions: Section 2.1

1. Problem 1: See the Maple worksheet to get the direction field. You should see that it looks like all solutions are approaching some curve (maybe a line?) as $t \rightarrow \infty$. To be more analytic, let us solve the DE using the Method of Integrating Factors.

$$y' + 3y = t + e^{-2t} \Rightarrow e^{3t}(y' + 3y) = e^{3t}(t + e^{-2t}) \Rightarrow (e^{3t}y(t))' = te^{3t} + e^t$$

Integrate both sides *Hint:* We need to use “integration by parts” to integrate te^{3t} . Using a table as in class:

+	t	e^{3t}
−	1	$(1/3)e^{3t}$
+	0	$(1/9)e^{3t}$

$$\Rightarrow \int te^{3t} dt = \frac{1}{3}e^{3t} - \frac{1}{9}e^{3t}$$

Putting it all together,

$$e^{3t}y(t) = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + C$$

so that

$$y(t) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{e^{-2t}} + \frac{C}{e^{3t}}$$

Notice that the last two terms go to zero as $t \rightarrow \infty$, so we see that $y(t)$ does approach a line:

$$\frac{1}{3}t - \frac{1}{9}$$

as $t \rightarrow \infty$.

2. Problem 3: See Maple for the direction field. Very similar situation to Problem 1. Let's go ahead and solve:

$$y' + y = te^{-t} + 1$$

Multiply both sides by $e^{\int p(t) dt} = e^t$:

$$e^t(y' + y) = t + e^t \Rightarrow (e^ty(t))' = t + e^t$$

Integrate both sides:

$$e^ty(t) = \frac{1}{2}t^2 + e^t + C \Rightarrow y(t) = \frac{1}{2}t^2e^{-t} + 1 + Ce^{-t}$$

This could be written as:

$$y(t) = 1 + \frac{t^2}{2e^t} + \frac{C}{e^t}$$

so that it is clear that, as $t \rightarrow \infty$, $y(t) \rightarrow 1$, which we also see in the direction field.

3. (Extra Practice, not assigned) Problem 11: See Maple for the direction field, where it looks like all solutions are approaching some periodic function as $t \rightarrow \infty$. Let's solve it:

$$y' + y = 5 \sin(2t)$$

As in the last exercise, multiply both sides by e^t :

$$e^t(y' + y) = 5e^t \sin(2t) \quad \Rightarrow \quad (e^t y(t))' = 5e^t \sin(2t)$$

To integrate the right-hand-side of this equation, we will need to use integration by parts twice. In tabular form:

+	e^t	$\sin(2t)$
-	e^t	$-(1/2) \cos(2t)$
+	e^t	$-(1/4) \sin(2t)$

$$\Rightarrow \int e^t \sin(2t) dt = -\frac{1}{2}e^t \cos(2t) + \frac{1}{4}e^t \sin(2t) - \frac{1}{4} \int e^t \sin(2t) dt$$

Add the last integral to the left:

$$\frac{5}{4} \int e^t \sin(2t) dt = -\frac{1}{2}e^t \cos(2t) + \frac{1}{4}e^t \sin(2t)$$

so that:

$$\int e^t \sin(2t) dt = -\frac{2}{5}e^t \cos(2t) + \frac{1}{5}e^t \sin(2t) + C_1$$

Going back to the differential equation,

$$e^t y(t) = -2e^t \cos(2t) + e^t \sin(2t) + C_2$$

so that the general solution is:

$$y(t) = -2 \cos(2t) + \sin(2t) + C_2 e^{-t}$$

We see that, as $t \rightarrow \infty$, $y(t)$ does indeed go to a periodic function.

In problems 13, 15, 16, solve the IVP. For these problems, I will leave the details out, but I will give the integrating factor. Be sure to ask in class if you're not sure how to solve them!

4. Problem 13: (You'll need to integrate by parts!)

$$y' - y = 2te^{2t} \quad e^{\int p(t) dt} = e^{-t}$$

$$y(t) = e^{2t}(2t - 2) + 3e^t$$

5. Problem 15:

$$ty' + 2y = t^2 - t + 1$$

Be sure to put in standard form before solving:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \quad e^{\int p(t) dt} = t^2$$

and

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

6. Problem 16: In this problem, the integrating factor is again t^2 :

$$y' + \frac{2}{t} \cdot y = \frac{\cos(t)}{t^2} \quad \Rightarrow \quad y(t) = \frac{\sin(t)}{t^2}$$

7. Problem 30: Solve the IVP:

$$y' - y = 1 + 3 \sin(t) \quad y(0) = y_0$$

The integrating factor is e^{-t} , so that we get:

$$(ye^{-t})' = e^{-t} + 3e^{-t} \sin(t)$$

Integrate both sides. The term on the far right side of the equation requires integration by parts using a table (differentiate the middle column, antidifferentiate the last column):

$$\int e^{-t} \sin(t) dt \quad \Rightarrow \quad \begin{array}{|c|c|c|} \hline + & \sin(t) & e^{-t} \\ \hline - & \cos(t) & -e^{-t} \\ \hline + & -\sin(t) & e^{-t} \\ \hline \end{array}$$

so that

$$\int e^{-t} \sin(t) dt = -e^{-t} \cos(t) - e^{-t} \sin(t) - \int e^{-t} \sin(t) dt$$

Add the last integral to both sides and divide:

$$2 \int e^{-t} \sin(t) dt = -e^{-t} \cos(t) - e^{-t} \sin(t)$$

Therefore,

$$\int e^{-t} \sin(t) dt = -\frac{1}{2}e^{-t} (\cos(t) + \sin(t))$$

Going back to the DE, we have (remember to multiply by 3):

$$ye^{-t} = -e^{-t} - \frac{3}{2}e^{-t} (\cos(t) + \sin(t)) + C$$

so that

$$y = -1 - \frac{3}{2} (\cos(t) + \sin(t)) + Ce^t$$

Therefore, the general solution is (using the initial condition):

$$y(t) = -1 - \frac{3}{2}(\cos(t) + \sin(t)) + \left(\frac{5}{2} + y_0\right)e^t$$

To keep the solution finite (or bounded) as $t \rightarrow \infty$, we must find y_0 so that the exponential term drops out- This means that $y_0 = -5/2$.

Problems 34-36: We're going backwards from a desired solution to the differential equation. We notice a couple of things about our previous work. If we want our solution to approach a given function $g(t)$, then we might write the full solution as

$$y(t) = g(t) + Ce^{-t}$$

Then, $y' = g'(t) - Ce^{-t}$, and $y' + y = g(t) - g'(t)$ is the differential equation.

8. For Problem 34, we want $y(t) \rightarrow 3$ as $t \rightarrow \infty$. Here are two possible ways of proceeding:

- Suppose $y(t) = 3 + Ce^{-t}$ (so it looks a lot like the solutions we got for the previous HW problems). Then $y' = -Ce^{-t}$, and we see that:

$$y' + y = 3$$

(I'll leave the verification to you).

- As another possible approach, we could take:

$$y(t) = 3 + \frac{C}{t^2}$$

Now, $y' = -2C/t^3$. We see that if we take ty' and add it to $2y$, the terms with C cancel and we're left with 6. Therefore, the ODE is:

$$ty' + 2y = 6$$

(I'll leave the verification to you).

9. Problem 35 is similar. There are many ways of constructing such a differential equation- It's easiest to start with a desired solution.

If we would like $y(t) = 3 - t + Ce^{-3t}$, then $y' = -1 - 3Ce^{-3t}$, and:

$$y' + 3y = 8 - 3t$$