Solutions: Section 2.3

NOTE: Some *Maple* commands are listed here for the benefit of students who have taken or are taking the Calculus Lab (Maple is mathematical software on the computers in the Math Computer Lab). Often, problems can be solved as well on Wolfram Alpha, and some comments on that are included.

1. Problem 3: In this model of salt in a tank of water, let $Q(t)$ be the amount of salt (in pounds) at time t (measured in minutes). Then dQ/dt will be measured in pounds per minute.

   The rate in, at least initially:

   $$\frac{1}{2}\cdot\frac{\text{lbs}}{\text{gal}}\cdot 2\cdot\frac{\text{gal}}{\text{min}} = 1\ \text{lbs/min}$$

   The rate out, at least initially:

   $$2\frac{\text{gal}}{\text{min}}\cdot\frac{Q(t)\ \text{lbs}}{100\ \text{gal}} = \frac{1}{50}Q\ \text{lbs/min}$$

   The model equation, valid for $0 \le t < 10$:

   $$\frac{dQ}{dt} = 1 - \frac{1}{50}Q \qquad Q(0) = 0$$

   The initial condition is zero, since we started with fresh water. You can solve this either as a linear equation, or a separable equation. We give the solution here as if it were linear:

   $$Q' + \frac{1}{50}Q = 1 \Rightarrow \mathrm{e}^{(1/50)t}\left(Q' + \frac{1}{50}Q\right) = \mathrm{e}^{(1/50)t} \Rightarrow Q(t) = 50 + C\mathrm{e}^{-(1/50)t}$$

   With the initial condition,

   $$0 = 50 + C \Rightarrow C = -50$$

   The amount of salt in the tank at time $0 \le t < 10$ minutes is:

   $$Q(t) = 50 - 50\mathrm{e}^{-(1/50)t}$$

   At time $t = 10$, the dynamics change. Suddenly no salt comes in (but water still does). Now we have:

   $$Q' = -\frac{1}{50}Q \qquad \text{valid for } t \ge 10$$

   Note: We could have left this as $t > 10$, but it was convenient to do it this way- That's because the initial condition for this DE is where we ended with the last DE,

   $$Q(10) = 50 - 50\mathrm{e}^{-1/5} \approx 9.06343$$

For $t > 10$, the solution is: $Q(t) = Ae^{-(1/50)t}$. Putting in $Q(10) = 9.06343$,

$$9.06343 = Ae^{-1/5} \quad \Rightarrow \quad A \approx 11.07$$

Therefore, for $t > 10$,

$$Q(t) = 11.07e^{-(1/50)t}$$

And now substitute $t = 20$ to find the final amount of salt in the tank: $Q(20) \approx 7.421$ pounds.

2. Problem 5: As usual, let $Q(t)$ be the amount of salt (in ounces) at time t (measured in minutes). Then dQ/dt will be measured in ounces per minute.

   The rate in:

   $$\frac{1}{4}\left(1 + \frac{1}{2}\sin(t)\right) \cdot \frac{\text{oz}}{\text{gal}} \cdot 2 \cdot \frac{\text{gal}}{\text{min}} = \frac{1}{2}\left(1 + \frac{1}{2}\sin(t)\right)\frac{\text{oz}}{\text{min}}$$

   The rate out:

   $$2\frac{\text{gal}}{\text{min}} \cdot \frac{Q(t)\text{ lbs}}{100\text{ gal}} = \frac{1}{50}Q\text{ lbs/min}$$

   The model equation:

   $$\frac{dQ}{dt} = \frac{1}{2}\left(1 + \frac{1}{2}\sin(t)\right) - \frac{1}{50}Q \qquad Q(0) = 50$$

   The initial condition was that we started with 50 ozs of salt in a tank of 100 gallons.

   This is a linear differential equation, and the integrating factor is the same as computed earlier in Problem 3:

   $$e^{(1/50)t}\left(Q' + \frac{1}{50}Q\right) = \frac{1}{2}e^{(1/50)t} + \frac{1}{4}e^{(1/50)t}\sin(t)$$

   Note that we need to integrate by parts twice, so that:

   $$\int e^{(1/50)t}\sin(t)\,dt = -\frac{2500}{2501}e^{(-1/50)t}\cos(t) - \frac{50}{2501}e^{(-1/50)t}\sin(t)$$

   Now we write the full solution:

   $$Q(t) = 25 - \frac{625}{2501}\cos(t) + \frac{25}{5002}\sin(t) + \frac{63150}{2501}e^{-(1/50)t}$$

   SORRY about those constants! Ugly as this might be, it is now easy to see what kind of behavior we can expect as $t \to \infty$- The last term drops out, and the salt varies periodically about the constant 25 (ozs.).

   We can verify this in Maple:

```
DE:=diff(Q(t),t)+(1/50)*Q(t)=(1/2)+(1/4)*sin(t);
Q1:=dsolve({DE,Q(0)=50},Q(t));
plot(rhs(Q1),t=0..400,numpoints=1000);
```

Or in Wolfram Alpha:

```
solve Q'+(1/50)Q=(1/2)+(1/4)sin(t)
```

For some reason, Wolfram Alpha didn't like the initial condition...

3. Problem 9: In the absence of payments, the rate of change of our loan will increase proportionally to the current amount. That is, let $S(t)$ be the amount of money (measured in dollars) owed at time t (measured in years). With no payments,

$$\frac{dS}{dt} = rS$$

where r is the annual interest rate (annual because we are measuring t in years). By making "continuous payments", at a constant annual rate k:

$$\frac{dS}{dt} = rS - k$$

We can solve this generally, with $S(0) = S_0$:

$$S(t) = \frac{k}{r} + \left(S_0 - \frac{k}{r}\right) \mathrm{e}^{rt}$$

Putting in the values, $S(0) = 8000$, $r = 1/10$ and leaving k as an adjustable parameter,

$$S(t) = 10k - (8000 - 10k)\,\mathrm{e}^{(1/10)t}$$

We want to find the value of k so that our loan is paid off in three years, or $S(3) = 0$:

$$0 = 10k - (8000 - 10k)\mathrm{e}^{3/10} \qquad k \approx 3086.64$$

(Side remark: That's about \$8.42 per day) So over the three year period, we would pay:

$$3 \cdot 3086.64 = 9259.92$$

so the interest paid was about \$1259.92.

4. Problem 10: We are assuming continuous compounding, and we'll let S be the amount of the loan remaining after time t. Then our model is (Exercise 9, above):

$$\frac{dS}{dt} = rS - k$$

In this case, it may be easiest to solve in general terms first (also given above):

$$S(t) = \frac{k}{r} + \left(S_0 - \frac{k}{r}\right) \mathrm{e}^{rt}$$

With $r = 0.09$ and a payment rate of $k = (800)(12) = 9600$, and $k/r = 106,666.666...$, we want $S(20) = 0$.

$$106666.67 + (S_0 - 106666.67)\mathrm{e}^{(0.09)(20)} = 0 \quad \Rightarrow \quad S_0 \approx \$89,034.79$$

The interest paid will be the total amount we paid (continuously, at a rate of \$ 9600 per year for 20 years is 192000) minus the principal,

$$192000 - 89034.79 = 102965.21$$

5. Problem 12: We are given that:

$$Q' = -rQ \quad \Rightarrow \quad Q(t) = Q_0\mathrm{e}^{-rt}$$

And we are told that the half-life of Carbon-14 is 5730 years. That means that, if Q_0 is the initial amount, then:

$$\frac{1}{2}Q_0 = Q_0\mathrm{e}^{-r\cdot(5730)}$$

Divide both sides by Q_0, and solve for r:

$$r = \frac{\ln(1/2)}{-5730} = \frac{\ln(2)}{5730} \approx 0.00012097 = 1.2097 \times 10^{-4}$$

The general solution is:

$$Q(t) = Q_0\mathrm{e}^{(-1.2907\times 10^{-4})t}$$

We now think of Q_0 as some unknown (but fixed) amount, and let T be the time it takes to decrease Q_0 to 20% of the original amount. Then solve for T:

$$\frac{1}{5}Q_0 = Q_0\mathrm{e}^{(-1.2907\times 10^{-4})T}$$

This gives $T \approx 13,304.65$ years.

6. Problem 13: Let us parse out the problem:

   - The population of mosquitoes increases at a rate proportional to the current population...

     If $P(t)$ is the population at time t, so far this says

     $$\frac{dP}{dt} = kP \qquad \Rightarrow \qquad P(t) = P_0\mathrm{e}^{kt}$$

- ... and in the absence of other factors, the population doubles each week. If we measure t in days, this means that:

  $$2P_0 = P_0 \mathrm{e}^{7k} \qquad \Rightarrow \qquad k = \frac{\ln(2)}{7} \approx 0.9902 \text{ per day}$$

  Now, going back to our model: So far, without predation, the rate of change population at time t (in days) is:

  $$\frac{dP}{dt} = \frac{\ln(2)}{7} P$$

- There are 200,000 mosquitoes initially (Modeled as $P(0) = 200,000$), and predators eat 20,000 per day- This is a constant decrease:

  $$\frac{dP}{dt} = \frac{\ln(2)}{7} P - 20,000, \qquad P(0) = 200,000$$

  Solve this the usual way (either as a linear or separable equation),

  $$P(t) = \frac{20,000 \cdot 7}{\ln(2)} + \left(200,000 - \frac{20,000 \cdot 7}{\ln(2)}\right) \mathrm{e}^{-0.9902t}$$

- If t is measured in weeks, then things simplify a bit. In that case, $k = \ln(2)$ and:

  $$\frac{dP}{dt} = \ln(2)P - 140,000 \qquad P(0) = 200,000$$

  and the solution to the IVP is:

  $$P(t) = \frac{140,000}{\ln(2)} + \left(200,000 - \frac{140,000}{\ln(2)}\right) 2^{-t}$$

  In numerical form,

  $$P(t) = 201,977.31 - 19,77.31 \cdot 2^{-t}$$

  Solving for when $P(t) = 0$, we see that the solution is valid for $0 \leq t \leq 6.6745$ (weeks).

7. Problem 23:

   A Physics Note: If something is measured in pounds, it has the same units as mass times gravity, mg. Gravity in this problem will be measured as 32 feet per second squared. Given that the weight is 180 pounds, and gravity is 32, we can then compute the mass: $180/32 = 5.625$.

Going back to our model from Section 1.1:

$$ma = m\frac{dv}{dt} = mg - kv \qquad \Rightarrow \qquad \frac{dv}{dt} = g - \frac{k}{m}v$$

Before the parachute opens, $0 \leq t \leq 10$, we have:

$$\frac{dv}{dt} = -\frac{\frac{3}{4}}{\frac{180}{32}}v + 32 = -\frac{2}{15}v + 32, \qquad v(0) = 0$$

Solving for the velocity equation,

$$v(t) = 240 - 240\mathrm{e}^{-(2/15)t}$$

The speed when the parachute opens ($t = 10$) is $v(10) \approx 176.74$ feet per second.

We can now integrate velocity to find position, $s(t)$. Careful here! A quick analysis of our velocity equation says that the velocity *towards the ground* is positive. But, if we say that $s(0) = 5000$, our height will be increasing (since $v = s' > 0$). To compensate, we set $s(0) = -5000$:

$$s(t) = -6800 + 240t + 1800\mathrm{e}^{-(2/15)t}$$

So the position at $t = 10$ is $s(10) \approx -3925.53$, which we interpret as 3925.53 feet above the ground, so the skydiver has fallen (approximately) $5000 - 3925.53 = 1074.47$ feet

To answer the last two questions, we reformulate the velocity equation. To simplify things, we'll reset the clock to $t = 0$ (interpret as minutes past 10):

$$\frac{dv}{dt} = -\frac{12}{\frac{180}{32}}v + 32 = -\frac{32}{15}v + 32$$

The "equilibrium" is $v(t) = 15$ feet per second. Solve this equation to with an "initial velocity" of 176.74, and:

$$v(t) = 15 + 161.74\mathrm{e}^{-(32/15)t}$$

Integrate to find position, with "initial position" at -3925.53:

$$s(t) = 15t - 75.82\mathrm{e}^{-(32/15)t} - 3849.71$$

To solve for t so that $s = 0$, we will need a computer. Here are the commands to first get an estimate, then solve in Maple:

```
S:=15*t-75.82*exp(-(32/15)*t)-3849.71;
plot(S,t=0..260);
fsolve(S=0,t=250..260);
```

and we get that $t \approx 256.6473333$ seconds, or about 4.3 minutes.

And in Wolfram Alpha:

```
solve 15*t-75.82*exp(-(32/15)*t)-3849.71=0
```

and in the plot, we see $t \approx 256.647$

8. Problem 28: From what is given in the problem, we'll use our standard model:

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

with $g = 9.8$ meters per seconds squared, $k = 0.2 = \frac{1}{5}$, and $m = 0.25 = \frac{1}{4}$. Therefore,

$$\frac{dv}{dt} = 9.8 - \frac{4}{5}v$$

and, with the initial condition $v(0) = 0$, we have:

$$v(t) = 12.25 - 12.25\mathrm{e}^{-(4/5)t}$$

We can now get position at time t, with the initial position -30:

$$s(t) = -45.31 + 12.25t + 15.31\mathrm{e}^{-(4/5)t}$$

We can now answer the first question, with a little Maple. To find the velocity when the ball hits the ground, we need to find the time at which this happens. Set $s(t) = 0$ and solve for t. The Maple commands are:

```
S:=-45.31+12.25*t+15.31*exp(-(4/5)*t);
plot(S,t=0..5);
fsolve(S=0,t=3..4);
```

Or, in Wolfram Alpha:

```
solve -45.31+12.25*t+15.31*exp(-(4/5)*t)=0
```

From this, $t \approx 3.63$ seconds. Substitute this into velocity:

$$v(3.63) \approx 11.58$$

For part (b), we want the velocity to be no more than 10 meters per second (what is the maximum height from which the ball can be dropped)? First, look at the velocity:

$$v(t) = 12.25 - 12.25\mathrm{e}^{-(4/5)t}$$

This is an increasing function (look at the plot in Maple, or consider that the derivative is positive). Therefore, we will find the time it takes for the velocity to reach 10 meters per second.

$$v(t) = 10 \Rightarrow 12.25 - 12.25\mathrm{e}^{-(4/5)t} = 10$$

Solve for t and get about $t = 2.1182$.

Now, look at the height function, $s(t)$, where $s(0) = S_0$:

$$s(t) = 12.25t + 15.31\mathrm{e}^{-(4/5)t} + (S_0 - 15.31)$$

We want to find S_0 so that $s(2.1182) = 0$. Substitute this value of t in and solve for S_0. You should find that $S_0 \approx -13.45$.

For part (c), we will need to use Maple, but let's see how far we can go before we need it: First we'll need velocity and position in terms of k:

$$\frac{dv}{dt} = 9.8 - (4k)v \qquad v(0) = 0$$

so that the solution in terms of k is:

$$v(t) = \frac{9.8}{4k} - \frac{9.8}{4k}\mathrm{e}^{-4kt}$$

The position at time t (with $s(0) = -30$) is:

$$s(t) = \frac{9.8}{4k}t + \frac{9.8}{16k^2}\mathrm{e}^{-4kt} - \left(30 + \frac{9.8}{16k^2}\right)$$

We now need to solve for k so that, when the ball hits the ground, the velocity is no more than 10. Let $t*$ be the time when the ball hits the ground- It too depends on k. Therefore, we have two equations in two unknowns (the unknowns are k and t^*):

$$v(t^*) = 10 \quad \Rightarrow \quad \frac{9.8}{4k} - \frac{9.8}{4k}\mathrm{e}^{-4kt^*} = 10$$

and

$$s(t^*) = 0 \quad \Rightarrow \quad \frac{9.8}{4k}t^* + \frac{9.8}{16k^2}\mathrm{e}^{-4kt^*} - \left(30 + \frac{9.8}{16k^2}\right) = 0$$

To solve this system of equations in Maple, we'll first define them, then plot the curves (to solve these numerically, we'll need to give Maple an approximate solution). Once we see the point of intersection, then Maple will solve it:

```
Eqn1:=(9.8/(4*k))-(9.8/(4*k))*exp(-4*k*t)=10;
Eqn2:=(9.8/(4*k))*t+(9.8/(16*k^2))*exp(-4*k*t)-(30+(9.8/(16*k^2)))=0;
with(plots):
implicitplot({Eqn1,Eqn2},k=0.1..1,t=0.1..5);
fsolve({Eqn1,Eqn2},{k,t},{k=0.2..0.25,t=3..5});
```

Maple gives the solution as:

```
{k = .2394381624, t = 3.952304030}
```

So we conclude that, if $k \geq 0.2394$, then the ball will hit the ground with a velocity of at most 10 meters per second.

(This is a lot easier to do in Maple than on Wolfram Alpha- It's plotting capabilities are limited in the free online version).