SOLUTIONS Extra Practice: Section 2.5

1. Given the differential equation, identify if each is *linear* (L), *separable* (S), *autonomous* (A), *Bernoulli* (B), and/or *homogeneous* (H). Recall that any given DE may have multiple labels.

   (a) $\dfrac{dy}{dx} = \dfrac{x^3 - 2y}{x}$

   SOLUTION: Linear. *NOTE:* The definition of the Bernoulli DE could be stretched to cover linear as well: $y' + p(t)y = q(t)y^n$ with n either 0 or 1, but we usually will exclude these, and we will here as well.

   (b) $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

   SOLUTION: Homogeneous, with $y' = \frac{1+(y/x)}{1-(y/x)}$.

   (c) $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$

   SOLUTION: Linear and Separable. We would probably treat is as separable to actually solve it, though.

   (d) $\dfrac{dy}{dt} = \cos(y)$

   SOLUTION: Separable and Autonomous.

   (e) $\dfrac{dy}{dt} = \cos(t)$

   SOLUTION: Linear and Separable.

   (f) $t^2y' + 2ty - y^3 = 0$

   SOLUTION: Bernoulli.

2. Suppose we are given the differential equation:

$$y' = \sin(y)$$

   (a) True or False: The solution may be periodic.

   ANSWER: False. Because isoclines are horizontal lines, if the solution $y(t)$ is increasing through a particular y value at a given time, then the function can never decrease through that y value again (at any time).

   (b) What happens to the solution corresponding to $y(0) = 1$? How about $y(0) = 100$? (HINT: Do not solve!)

   ANSWER: If we plot the sine function, we will see that the all equilibria are of the form $y = k\pi$, for $k = 0, \pm 1, \pm 2$,... We also see that the odd multiples of π:

$$y(t) = \pi, 3\pi, 5\pi, \ldots$$

are attracting equilibria, and the others are all repelling. Therefore, if the solution starts at $y = 1$ (really for any given time), then as time goes on, $y(t) \to \pi$. Furthermore, $31\pi < 100 < 32\pi$, so if $y = 100$ for any given time, as time goes on, $y(t) \to 31\pi$.

3. Below are two direction fields (in the (t, y) plane). Find an autonomous differential equation, $y' = F(y)$, that is consistent with each one. Proceed by first sketching a consistent function for each direction field in the (y, y') plane.

   SOLUTION: From the equilibria and where y is increasing/decreasing, we should get polynomials: $y' = -y(y-1)$ and $y' = -(y-1)(y-2)^2$

4. Let $y' = y(y-1)$.

   (a) Give the general solution.

   SOLUTION: The equation is separable. To integrate, we'll need partial fractions:

$$\int \frac{1}{y(y-1)}\,dy = \int dt \quad \Rightarrow \quad \int \frac{-1}{y} + \frac{1}{y-1}\,dy = t + C \quad \Rightarrow \quad -\ln|y| + \ln|y-1| = t + C$$

   Continuing,

$$\ln\left|\frac{y-1}{y}\right| = t + C \quad \Rightarrow \quad \frac{y-1}{y} = Ae^t \quad \Rightarrow \quad y(t) = \frac{1}{1 + Be^t}$$

   (b) Plot the appropriate function in the (y, y') plane, and classify the equilibria as to stability.

   SOLUTION: Your curve is the parabola from the previous problem, but flipped. Therefore, $y = 0$ is stable and $y = 1$ is unstable.

   (c) Without going to the solution $y(t)$, find intervals on which $y(t)$ will be concave up and concave down.

   SOLUTION: Split the $y-$axis into four parts-

   - $y < 0$: In this region, $F(y) > 0$ and $dF/dy < 0$, so $y(t)$ will be Concave Down.
   - $0 < y < 1/2$: In this region, $F(y) < 0$ and $dF/dy < 0$, so $y(t)$ will be Concave Up.
   - $1/2 < y < 1$: In this region, $F(y) < 0$ and $dF/dy > 0$, so $y(t)$ will be Concave Down.
   - $y > 1$: In this region, both F and dF/dy are positive, so $y(t)$ will be Concave Up.

5. For each fraction, write down what the partial fraction expansion would be (but do not solve for the constants!):

(a) SOLUTION: The orders are right and we have all linear factors:

$$\frac{x^2-3x+1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

(b) SOLUTION: The orders are right and we have one repeated factor:

$$\frac{3x-1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

Alternate Solution: We can treat x^2 as an irreducible quadratic (and note that both solutions give the same thing).

$$\frac{3x-1}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1}$$

(c) SOLUTION: One linear, one irreducible quadratic:

$$\frac{3x-1}{(x+1)(x^2+2x+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+3}$$

Side Remark: Completing the square in the denominator,

$$x^2+2x+3 = (x^2+2x+1)+2 = (x+1)^2+2$$

will tell you quickly if the expression is reducible (you would be subtracting instead of adding 2), and you would have to do it to integrate the expression anyway.

(d) SOLUTION: One irreducible quadratic, one repeated irreducible quadratic:

$$\frac{3x-1}{(x^2+1)(x^2+4)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}$$