

Selected solutions, 2.6

NOTE: In Problems 1 and 3, we show two different ways of finding the underlying function f . Either way is fine.

1. Take $M = 2x + 3$ and $N = 2y - 2$. Then $M_y = N_x = 0$, and to find the solution, we can antidifferentiate M :

$$f(x, y) = \int M dx = x^2 + 3x + h_1(y)$$

We can differentiate this to see if we get N : $f_y = h_1'(y) = 2y - 2$. Therefore, $h_1(y) = y^2 - 2y$. Put it all together to get the solution:

$$x^2 + 3x + y^2 - 2y = C$$

3. $(3x^2 - 2xy + 2) dx + (6y^2 - x^2 + 3) dy = 0$

SOLUTION: If this is exact, this is of the form $f_x dx + f_y dy$ for some f . We use the test ($M_y = N_x$):

$$M_y = -2x \quad N_x = -2x$$

Now we try to reconstruct f . One way to do it is to integrate twice and compare:

$$f_x = 3x^2 - 2xy + 2 \Rightarrow f(x, y) = x^3 - x^2y + 2x + h_1(y)$$

Using the other function,

$$f_y = 6y^2 - x^2 + 3 \Rightarrow f(x, y) = 2y^3 - x^2y + 3y + h_2(x)$$

Now compare the two expressions to see that $f(x, y) = x^3 + 2y^3 - x^2y + 2x + 3y$ and the general solution is:

$$x^3 + 2y^3 - x^2y + 2x + 3y = C$$

4. Solving it as written, we should get $x^2y^2 + 2xy = C$, but did you notice that you could factor $2xy + 2$ out of M and N (if not, that's OK). In that case, the equation becomes simpler- In fact, we end up $xy = C$ as the solution.
13. You should get that the general solution is

$$x^2 - xy + y^2 = C$$

and that the initial condition yields $c = 7$. In this case, one could solve the specific solution for y by completing the square in y (or you could use the quadratic formula in y):

$$\begin{aligned} y^2 - xy = 7 - x^2 &\Rightarrow y^2 - xy + \frac{x^2}{4} = 7 - \frac{3x^2}{4} = \\ \left(y - \frac{x}{2}\right)^2 &= \frac{28 - 3x^2}{4} \Rightarrow y = \frac{x + \sqrt{28 - 3x^2}}{2} \end{aligned}$$

(Positive root to match the IC) so the solution is valid as long as $3x^2 \leq 28$.

17. There is a small (but important) typo. The function ψ (the symbol ψ is read like the beginning of “psychology”):

$$\psi(x, y) = \int_{x_0}^x M(s, y_0) ds + \int_{y_0}^y N(x_0, t) dt$$

Having said that, I believe this question is suggesting something that is incorrect. If you have the time and interest, you might try working out a specific example to see what happens.

18. In the case that M is a function of x alone, and N is a function of y alone, then $M_y = N_x = 0$.
19. Multiply by the integrating factor so that the new DE is exact:

$$\frac{x^2 y^3}{x y^3} + \frac{x(1+y^2)}{x y^3} y' = 0 \quad \Rightarrow \quad x + \frac{1+y^2}{y^3} y' = 0$$

This is like Exercise 18- This is a separable DE, but we'll solve it as an exact equation:

$$M(x, y) = x \quad \Rightarrow \quad f(x, y) = \frac{1}{2} x^2 + h_1(y)$$

$$N(x, y) = \frac{1}{y^3} + \frac{1}{y} \quad \Rightarrow \quad f(x, y) = -\frac{1}{2y^2} + \ln|y| + h_2(x)$$

Put these together:

$$\frac{1}{2} x^2 - \frac{1}{2y^2} + \ln|y| = C$$

(The text multiplied everything by 2)

22. With the given integrating factor, we write:

$$(x+2)xe^x \sin(y) + x^2 e^x \cos(y) y' = 0$$

Now this is exact, since $M_y = (x^2 + 2x)e^x \cos(y)$, and so is N_x (remember to use the product rule). Now to integrate, the author has been kind to us- Remember, we can choose which integral to do- Either M with respect to x (a little messy) or N with respect to y (easy!):

$$f(x, y) = \int N dy = x^2 e^x \sin(y) + h_1(x)$$

and now determine if there is a function $h_1(x)$ by comparing f_x to M :

$$f_x = \sin(y) (2xe^x + x^2 e^x) + h_1'(x) \quad \Rightarrow \quad h_1'(x) = 0$$

and the solution is:

$$x^2 e^x \sin(y) = C$$