Variation of Parameters (3.6)

Summary

Given $y'' + p(t)y' + q(t)y = g(t)$, we assume

$$y_p = u_1 y_1 + u_2 y_2$$

Where y_1, y_2 solve the homogeneous DE. Then u_1, u_2 satisfy

$$\begin{array}{ll} u_1' y_1 + u_2' y_2 & = 0 \\ u_1' y_1' + u_2' y_2' & = g(t) \end{array} \quad \Rightarrow u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

Examples

1. Use Variation of Parameters to find the general solution:

$$4y'' - 4y' - 8y = 8\mathrm{e}^{-t}$$

In standard form,

$$y'' - y' - 2y = 2\mathrm{e}^{-t}$$

so that

$$r = -1, 2 \quad \Rightarrow \quad y_1 = \mathrm{e}^{2t} \qquad y_2 = \mathrm{e}^{-t} \qquad g(t) = 2\mathrm{e}^{-t}$$

This was the error from class- We need to write the DE in standard form to determine $g(t)$... This error made an important point!

We need $W(y_1, y_2)$:

$$W = \begin{vmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-t} \\ 2\mathrm{e}^{2t} & -\mathrm{e}^{-t} \end{vmatrix} = -3\mathrm{e}^{t}$$

Substituting into the formulas,

$$u_1' = \frac{\begin{vmatrix} 0 & \mathrm{e}^{-t} \\ 2\mathrm{e}^{-t} & -\mathrm{e}^{-t} \end{vmatrix}}{-3\mathrm{e}^{t}} = \frac{-2\mathrm{e}^{-2t}}{-3\mathrm{e}^{t}} = \frac{2}{3}\mathrm{e}^{-3t} \qquad u_2' = \frac{\begin{vmatrix} \mathrm{e}^{2t} & 0 \\ 2\mathrm{e}^{2t} & 2\mathrm{e}^{-t} \end{vmatrix}}{-3\mathrm{e}^{t}} = \frac{2\mathrm{e}^{t}}{-3\mathrm{e}^{t}} = -\frac{2}{3}$$

Therefore, $u_1 = -\frac{2}{9}\mathrm{e}^{-3t} + C_1$ and $u_2(t) = -\frac{2}{3}t + C_2$, so that

$$y_p = -\frac{2}{9}\mathrm{e}^{2t} + C_1\mathrm{e}^{2t} - \frac{2}{3}t\mathrm{e}^{-t} + C_2\mathrm{e}^{-t}$$

From which we observe that the only term that is not part of y_h is:

$$y_p = -\frac{2}{3}t\mathrm{e}^{-t}$$

so the full solution is

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2y} - \frac{2}{3}t\mathrm{e}^{-t}$$

We'll verify this using the Method of Undetermined Coefficients. The homogeneous part of the equation remains the same, but we guess that

$$y_p = At\mathrm{e}^{-t}$$

since $A\mathrm{e}^{-t}$ is part of the homogeneous solution. Substituting this into the DE, we get:

$$\begin{array}{rcll} -8y_p & = & & -8At\mathrm{e}^{-t} \\ -4y_p' & = & (-4A & +4At)\mathrm{e}^{-t} \\ 4y_p'' & = & (-8A & +4At)\mathrm{e}^{-t} \\ \hline 8\mathrm{e}^{-t} & = & (-12A & +0t)\mathrm{e}^{-t} \end{array} \quad \Rightarrow \quad A = -\frac{2}{3}$$

so we get the same solution as before.

2. Use any method to find the general solution:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1+t^2} + 3\mathrm{e}^t$$

SOLUTION: First, the solution to the characteristic equation is $r = 1, 1$

Now, we will use both Variation of Parameters **and** Method of Undetermined Coefficients. First, we'll find the particular solution if $g(t) = \mathrm{e}^t/(1+t^2)$, then we'll find the other one.

Here we go for Variation of Parameters:

$$y_1 = \mathrm{e}^t \qquad y_2 = t\mathrm{e}^t \qquad g(t) = \frac{\mathrm{e}^t}{1+t^2} \qquad W = \mathrm{e}^{2t}$$

Therefore

$$u_1' = \frac{-t\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{-t}{1+t^2} \quad \Rightarrow \quad u_1 = -\frac{1}{2}\ln(1+t^2)$$

$$u_2' = \frac{\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{1}{1+t^2} \quad \Rightarrow \quad u_2 = \tan^{-1}(t)$$

NOTE: We do not need the constants, as this will just bring in components from the homogeneous part of the solution.

Therefore, the particular part of the solution is:

$$y_{p_1}(t) = -\frac{1}{2}\mathrm{e}^t\ln(1+t^2) + t\mathrm{e}^t\tan^{-1}(t)$$

Next, we solve the other part. We guess $A\mathrm{e}^t$, but we'll need to multiply by t^2:

$$y_{p_2} = At^2\mathrm{e}^t \qquad y_{p_2}' = (At^2 + 2At)\mathrm{e}^t \qquad y_{p_2}'' = (At^2 + 4At + 2A)\mathrm{e}^t$$

We substitute into the DE and solve for the coefficients:

$$\begin{array}{rllll} y_p & = (At^2 & & &)\mathrm{e}^t \\ -2y_p' & = (-2At^2 & -4At & &)\mathrm{e}^t \\ y_p'' & = (At^2 & +4At & +2A &)\mathrm{e}^t \\ \hline 3\mathrm{e}^t & = (0 & 0 & 2A &)\mathrm{e}^t \end{array} \Rightarrow A = \frac{3}{2}$$

Therefore, the overall solution is:

$$y(t) = \mathrm{e}^t \left(C_1 + C_2 t + \frac{3}{2} t^2 \right) - \frac{1}{2} \mathrm{e}^t \ln(1 + t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

3. Use the Variation of Parameters to solve

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3 \qquad y_1(t) = t \quad y_2(t) = t\mathrm{e}^t$$

SOLUTION: $g(t) = 2t$ and $W(y_1, y_2) = t^2\mathrm{e}^t$, so

$$u_1' = \frac{-2t^2\mathrm{e}^t}{t^2\mathrm{e}^t} = -2 \quad \Rightarrow \quad u_1 = -2t$$

Similarly,

$$u_2' = \frac{2t^2}{t^2\mathrm{e}^t} = 2\mathrm{e}^{-t} \quad \Rightarrow \quad u_2 = -2\mathrm{e}^{-t}$$

so

$$y_p = (-2t)(t) + (-2\mathrm{e}^{-t})(t\mathrm{e}^t) = -2t^2 - 2t \quad \Rightarrow y_p(t) = -2t^2$$

Summary of Techniques

1. We had two techniques for finding a fundamental set of solutions to a homogeneous linear second order differential equation (with possibly non-constant coefficients):

   - Reduction of Order: Assume $y_2 = v(t)y_1(t)$, and substitute into the DE. You'll get a differential equation in v'' and v' (so first order in v') that we can then solve.
   - Wronskian: We can compute the Wronskian in two ways- Abel's Theorem and the usual method. This gives a first order DE in y_2 (given y_1) that we can solve.

   The second method is probably easier to use in many instances.

2. We had two techniques for finding the particular solution to a non-homogeneous second order linear DE (with forcing function $g(t)$):

   - Method of Undetermined Coefficients ($g(t)$ has to be of a certain type).
   - Variation of Parameters (This section).

   The second method is more general than the first, but can be more difficult to implement (because of the integrals).