Mass-Spring Model (3.7)

Suppose we have a spring hanging down from the ceiling with natural length l. If I attach a mass of m kg, the spring will lengthen to it's new length. Call this L (units past l). Hooke's Law states that the restorative force of the spring is proportional to the length it is stretched,

$$F_{\text{restore}} = -kL$$

When the mass is resting at the end of the spring, there is no motion, so that we have an equilibrium:

$$mg - kL = 0$$

We will define $u(t)$ to be the displacement of the spring from it's resting state (positive is down). Then, the text shows us the DE describing the motion of the mass is given by the following IVP:

$$mu'' + \gamma u' + ku = F(t) \qquad u(0) = u_0 \quad u'(0) = v_0$$

where m is the mass, γ is the damping constant (this is analogous to our falling body with air resistance from Chapter 2), and k is the spring constant. We call $F(t)$ the external forcing function.

NOTE: Although the text likes to mix units, on the exam I will not do that- I will be consistent in either (kg, seconds, meters) or we may use lbs, feet and seconds (and remember that weight is mg).

We will solve these using the techniques we have already learned, but we want to consider the following: If m, γ, k are all non-negative, what kinds of solutions will we get?

- $\gamma^2 - 4mk > 0$: In this case, notice that $0 < \gamma^2 - 4mk < \gamma^2$, so

  $$-\gamma \pm \sqrt{\gamma^2 - 4mk} < 0$$

  This implies that both r_1, r_2 will be negative. Therefore, $y_h(t) \to 0$ for all initial conditions.

- $\gamma^2 - 4mk = 0$. In this case,

  $$r = -\frac{\gamma}{2m} < 0$$

  so that $\mathrm{e}^{rt}(C_1 + C_2 t) \to 0$ for all initial conditions.

- $\gamma^2 - 4mk < 0$: In this case,

  $$r = -\frac{\gamma}{2m} \pm \left(\frac{\sqrt{4mk - \gamma^2}}{2m}\right) i = -\alpha \pm \beta i$$

  so that

  $$y_h(t) = \mathrm{e}^{-\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right) \to 0$$

  for all initial conditions.

From the Method of Undetermined Coefficients, if F is a sinusoidal function, then the particular part of the solution will **not** go to zero as $t \to \infty$.