Summary- Elements of 7.3 and 7.5

1. Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that
$$A\mathbf{v} = \lambda\mathbf{v}$$
then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector.

2. Eigenvectors are not unique. That is, if $\mathbf{v}$ is an eigenvector for A, so is $k\mathbf{v}$ (prove it!).

3. If you're starting to compute them for the first time, start with the original definition and work through to the system:
$$A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow \begin{array}{lll} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \Leftrightarrow \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array}$$
This system has a non-trivial solution for v_1, v_2 only if the determinant of coefficients is 0:
$$\left| \begin{array}{cc} a-\lambda & b \\ c & d-\lambda \end{array} \right| = 0$$
And this is the **characteristic equation**. We solve this for the eigenvalues:
$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \Leftrightarrow \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$$
where $\text{Tr}(A)$ is the trace of A (which we defined as $a+d$). We see from the quadratic formula that the solution depends on the discriminant (We'll continue this on day 2).

4. Connecting to systems of differential equations, given
$$\mathbf{x}' = A\mathbf{x}$$
If we make the ansatz:
$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{rt}\left[\begin{array}{c} v_1 \\ v_2 \end{array} \right] = \left[\begin{array}{c} \mathrm{e}^{\lambda t}v_1 \\ \mathrm{e}^{\lambda t}v_2 \end{array} \right]$$
then we saw that $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

5. If there are two distinct real eigenvalues, the solution to the system $\mathbf{x}' = A\mathbf{x}$ is given by:
$$\mathbf{x} = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

6. The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$. Expanding the ideas from Chapter 2, we can classify an equilibrium solution in many ways. Last time, we classified the origin as a *sink* (if the eigenvalues are both negative), a *source* (if both eigenvalues are positive), or a *saddle* (if one eigenvalue is positive and one is negative), and we drew some graphs of the *phase plane*.