Selected Solutions, Section 3.3 (Complex)

7. Solving the characteristic equation, $r = 1 \pm i$. Therefore,

$$y = \mathrm{e}^t \left(C_1 \cos(t) + C_2 \sin(t)\right)$$

8. Solving the characteristic equation, $r = 1 \pm \sqrt{5}\, i$. Therefore,

$$y = \mathrm{e}^t \left(C_1 \cos(\sqrt{5}\, t) + C_2 \sin(\sqrt{5}\, t)\right)$$

9. Solving the characteristic equation, $r = 2, -4$. Therefore,

$$y = C_1 \mathrm{e}^{2t} + C_2 \mathrm{e}^{-4t}$$

(Similarly, we solve 10-15).

25. For this one, try to go as far as you can without assistance from the computer:
Let $y'' + 2y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = \alpha \geq 0$.

(a) Solve it: The characteristic equation is $r^2 + 2r + 6 = 0$. Therefore, $r = -1 \pm \sqrt{5}\, i$ and the general solution is:

$$y(t) = \mathrm{e}^{-t} \left(C_1 \cos(\sqrt{5}\, t) + C_2 \sin(\sqrt{5}\, t)\right)$$

With the initial conditions, find that $C_1 = 2$ and $C_2 = \frac{\alpha+2}{\sqrt{5}}$, so that the solution to the IVP is:

$$y(t) = \mathrm{e}^{-t} \left(2\cos(\sqrt{5}\, t) + \frac{\alpha + 2}{\sqrt{5}} \sin(\sqrt{5}\, t)\right)$$

(b) Find α so that $y(1) = 0$: Algebraically, we get:

$$-2\sqrt{5}\, \frac{\cos(\sqrt{5})}{\sin(\sqrt{5})} - 2 = \alpha$$

(c) Find the smallest value of $t > 0$ so that $y(t) = 0$. We'll write our answer as a function of α.
Algebraically, solving this:

$$0 = \mathrm{e}^{-t} \left(2\cos(\sqrt{5}\, t) + \frac{\alpha + 2}{\sqrt{5}} \sin(\sqrt{5}\, t)\right)$$

will get us to:

$$\frac{-2\cos(\sqrt{5}\, t)}{\sin(\sqrt{5}\, t)} = \frac{\alpha + 2}{\sqrt{5}}$$

We want to solve this for t, remembering that we want the smallest $t > 0$. I think it is easiest to work with the tangent rather than the cotangent,

$$\tan(\sqrt{5}\, t) = \frac{-2\sqrt{5}}{\alpha + 2}$$

From here, we're meant to use technology, so it's OK to stop here.

27. Straightforward computation. Recall what we said in class- If we let $r = \lambda + i\mu$, then

$$y_1 = \mathrm{Re}(\mathrm{e}^{rt}) = \mathrm{Re}(\mathrm{e}^{\lambda t + (\mu t)i}) = \mathrm{e}^{\lambda t}\cos(\mu t)$$

and

$$y_2 = \mathrm{Im}(\mathrm{e}^{rt}) = \mathrm{Im}(\mathrm{e}^{\lambda t + (\mu t)i}) = \mathrm{e}^{\lambda t}\sin(\mu t)$$

then $W(y_1, y_2) = \mu \mathrm{e}^{2\lambda t} \neq 0$, so y_1, y_2 will form a fundamental set of solutions to our second order linear homogeneous DE with constant coefficients (in the case where we have complex roots).

35, 37 The important part of these problems was to be able to do the chain rule to find expressions for the derivatives. We will use the convention that a dot means derivative with respect to time: $\dot{y} = dy/dt$.

Continuing, one way to do these, with $x = \ln(t)$ (so $dx = (1/t)\,dt$):

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = t\dot{y}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(t\dot{y}\right)\frac{dt}{dx} = t^2\ddot{y} + t\dot{y}$$

Now, for example, in 37:

$$t^2\ddot{y} + 3t\dot{y} + \frac{5}{4}y = 0$$

can be written as:

$$(t^2\ddot{y} + t\dot{y}) + 2(t\dot{y}) + \frac{5}{4}y = 0$$

And with our substitution:

$$y'' + 2y' + \frac{5}{4}y = 0 \quad \Rightarrow \quad r = -1 \pm \frac{1}{2}\,i$$

so that

$$y(x) = \mathrm{e}^{-x}\left(C_1\cos\left(x/2\right) + C_2\sin\left(x/2\right)\right)$$

then backsubstitute $x = \ln(t)$ to get y in terms of t.

Interesting side remark: Some of you found that using an ansatz: $y = t^r$ gives us the same characteristic equation! For example, in 37, using this ansatz gives:

$$r(r-1)t^r + 3rt^r + \frac{5}{4}t^r = 0 \qquad \Rightarrow \qquad r^2 + 2r + \frac{5}{4} = 0$$

Thus, $r = -1 \pm \frac{1}{2}i$. How should we interpret this? We could take

$$t^r = t^{-1+i/2} = (t^{-1})(t^{i/2})$$

To understand what it means to raise t to a complex number power, we use the identity:

$$A = \mathrm{e}^{\ln(A)} \quad \Rightarrow \quad t^{i/2} = \mathrm{e}^{\ln(t^{i/2})} = \mathrm{e}^{(\ln(t)/2)\,i)} = \cos(\ln(t)/2) + i\sin(\ln(t)/2)$$

so that the answer we got previously is $C_1\mathrm{Re}(t^r) + C_2\mathrm{Im}(t^r)$.