

Selected Problems from Section 3.6 (Variation of Params)

1. Solve: $y'' - 5y' + 6y = 2e^t$

NOTE: If we were not specifically told to use Variation of Parameters, we should choose Method of Undetermined Coefficients to solve this problem. For this exercise, we're asked to do both. The homogeneous part has $y_1 = e^{2t}$ and $y_2 = e^{3t}$ for its fundamental set. Therefore,

- Compute the Wronskian: $W(e^{2t}, e^{3t}) = e^{5t}$. Therefore,

$$u_1' = \frac{-2e^{4t}}{e^{5t}} = -2e^{-t} \Rightarrow u_1 = 2e^{-t}$$

$$u_2' = \frac{2e^{3t}}{e^{5t}} = 2e^{-2t} \Rightarrow u_2 = -e^{-2t}$$

Therefore,

$$y_p = u_1 y_1 + u_2 y_2 = 2e^t - e^t = e^t$$

- Verify the previous answer using Method of Undetermined Coefficients, with $y_p = Ae^t$. Note that $y = y' = y''$, so substitution yields:

$$e^t(A - 5A + 6A) = 2e^t \Rightarrow A = 1$$

$$\text{so } y_p(t) = e^t.$$

2. Same idea as Exercise 1. If we take

$$y_1 = e^{2t} \quad y_2 = e^{-t}$$

Then: $W(y_1, y_2) = -3e^t$, and

$$u_1' = \frac{2}{3}e^{-3t} \Rightarrow u_1 = -\frac{2}{9}e^{-3t}$$

and

$$u_2' = -\frac{2}{3} \Rightarrow u_2 = -\frac{2}{3}t$$

Substitution into $y_p = u_1 y_1 + u_2 y_2$ gives

$$y_p(t) = -\frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}$$

Since e^{-t} is part of y_h , we can simplify y_p :

$$y_p(t) = -\frac{2}{3}te^{-t}$$

And, using Method of Undetermined Coefficients, we would initially guess $y_p = Ae^{-t}$. Since that is part of y_h , we multiply by t so that

$$y_p = Ate^{-t}$$

And you should be able to verify that substitution into the DE yields the same answer as before.

6. The solution to the characteristic equation is $r = \pm 3i$, so the solution to the homogeneous part of the equation is

$$y_h = C_1 \cos(3t) + C_2 \sin(3t)$$

The Wronskian is 3. Setting up the formulas and integrals, we get:

$$u_1 = -3 \int \frac{\sin(3t)}{\cos^2(3t)} dt$$

And use u, du substitution to get $u_1(t) = -\csc(3t)$. For the other function,

$$u_2 = 3 \int \frac{1}{\cos(3t)} dt$$

The integral of the secant is a bit tricky, so I would not give you that integral on an in-class exam. The integral simplifies to: $u_2 = \ln |\sec(3t) + \tan(3t)|$. Finish up by constructing y_p , then put everything together.

9. If we take

$$y_1 = \cos(t/2) \quad y_2 = \sin(t/2)$$

then $W = 1/2$ and (writing the DE in standard form): $g(t) = \sec(t/2)/2$. Setting up the integrals, we get:

$$u_1 = - \int \frac{\sin(t/2)}{\cos(t/2)} dt = 2 \ln |\cos(t/2)|$$

(which we integrated using u, du substitution with $(u = \cos(t/2))$), and

$$u_2 = \int 1 dt = t$$

Therefore, the particular part of the solution is:

$$y_p = 2 \cos(t/2) \ln |\cos(t/2)| + t \sin(t/2)$$

To finish, write down $y_h + y_p$.

12. If we take $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$, then $W = 2$, and

$$u_1 = -\frac{1}{2} \int g(t) \sin(2t) dt \quad u_2 = \frac{1}{2} \int g(t) \cos(2t) dt$$

We can then write the particular part of the solution as $u_1 y_1 + u_2 y_2$. The textbook uses a trig identity to get the answer in the back of the text (this simplification would not be required on an exam/quiz).

13. Taking $p = 0$, $q = -2/t^2$, and $g(t) = 3 - 1/t^2$, we get a Wronskian that is -3 which leads us to:

$$u_1 = \int \frac{-3t^{-1} + t^{-3}}{-3} dt = \ln|t| + \frac{1}{6}t^{-2}$$

and

$$u_2 = \int \frac{3t^2 - 1}{-3} dt = -\frac{1}{3}t^3 + \frac{1}{3}t$$

Then form $y_p = u_1 y_1 + u_2 y_2$.

15. We won't verify that y_1, y_2 are solutions to the homogeneous equation, but you should. Assuming that they are (remember to put the equation in standard form by dividing by t), we compute the Wronskian and use the formulas from Variation of Parameters:

$$y_1 = 1 + t \quad y_2 = e^t \quad g(t) = te^{2t} \quad W = te^t$$

$$u_1' = \frac{-e^t te^{2t}}{te^t} = -e^{2t} \Rightarrow u_1(t) = -\frac{1}{2}e^{2t}$$

For u_2 , we'll integrate by parts:

$$u_2' = \frac{(1+t)te^{2t}}{te^t} = (1+t)e^t \Rightarrow u_2(t) = te^t$$

The particular solution is $u_1 y_1 + u_2 y_2$ which simplifies to

$$y_p = \frac{1}{2}e^{2t}(t-1)$$

29. This problem is good for practice using Reduction of Order: The DE in standard form is:

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 4$$

so that $y_2 = tv(t) = tv$, and substitution yields Equation (ii) in Exercise 28:

$$tv'' = 4 \quad v'' = \frac{4}{t} \Rightarrow v' = 4 \ln(t) + C_1$$

Therefore,

$$v(t) = 4 \int \ln(t) dt + C_1 t + C_2$$

The integral of $\ln(t)$ is done by “integration by parts” with $u = \ln(t)$ and $dv = dt$ (I would either not give you this integral or give you a hint if this were on an exam). Our final version of v is then:

$$v = 4t \ln(t) - 4t + C_1 t + C_2$$

Since $y_1 = t$, we can take $vy_1 = 4t^2 \ln(t) + C_3 t^2$, so that $y_2 = t^2$ and $y_p = 4t^2 \ln(t)$.