Summary of 5.4

Only the first part of Section 5.4 will be on the exam (it will exclude discussion of regular singular points from 5.4). Therefore, only questions similar to 1, 3, 5 and 10 would be on the exam. These come from the following:

Given the **Euler Equation**

$$x^2y'' + \alpha xy' + \beta y = 0$$

We use the ansatz $y = x^r$ and substitute into the DE, giving us the characteristic equation:

$$(r(r-1) + \alpha r + \beta) = 0$$

Solve for r using the quadratic formula. The discriminant is:

$$(\alpha - 1)^2 - 4\beta$$

Three cases:

- If the discriminant is positive, we have two real, distinct roots. The general solution is:
$$y = C_1x^{r_1} + C_2x^{r_2}$$
- If the discriminant is zero:
$$y = x^r(C_1 + C_2\ln(x))$$
- If the discriminant is negative, $r = \lambda \pm \mu i$, and
$$y = C_1x^\lambda\cos(\mu\ln(x)) + C_2x^\lambda\sin(\mu\, ln(x))$$

These are assuming that $x > 0$. If x could be negative, x should be replaced by $|x|$ in each instance. Here are the exercises:

1.
$$x^2y'' + 4xy' + 2y = 0$$
The characteristic equation is:
$$r(r-1) + 4r + 2 = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0$$
So the general solution is:
$$y = C_1x^{-1} + C_2x^{-2}$$

3.
$$x^2y'' - 3xy' + 4y = 0$$
As before, go to the characteristic equation:
$$r(r-1) - 3r + 4 = 0 \quad \Rightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad (r-2)^2 = 0$$
The solution is:
$$y = x^2(C_1 + C_2\ln|x|)$$

10. This one is a bit different. You can let $w = x - 2$, then back-substitute at the end.

$$w^2y'' + 5wy' + 8y = 0 \quad \Rightarrow \quad r(r-1) + 5r + 8 = 0 \quad \Rightarrow \quad r^2 + 4r + 8 = 0$$

Complete the square (or use the quadratic formula):

$$(r^2 + 4r + 4) + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = -4 \quad \Rightarrow \quad r = -2 \pm 2i$$

Therefore, the solution is:

$$y = x^{-2}\left(C_1 \cos(2\ln|x-2|) + C_2 \sin(2\ln|x-2|)\right)$$

You should try Exercise 8 as well.