

Help on section 6.2

The main ideas:

- Take the Laplace transform of a DE.
- Solve it for $Y(s)$ (the transform of $y(t)$).
- Invert the transform- This may involve partial fractions and/or completing the square.

The first set of exercises focuses on this last part, where partial fractions and/or completing the square will be performed in order to use the table to do the inverse transform.

2. Table entry 11.

4. Use Partial Fractions:

$$\frac{3s}{s^2 - s - 6} = \frac{A}{s + 2} + \frac{B}{s - 3}$$

so that

$$A(s - 3) + B(s + 2) = 3s$$

If $s = 3$, then $B = 9/5$. If $s = -2$, then $A = 6/5$. Now,

$$\mathcal{L}^{-1}\left(\frac{6}{5} \frac{1}{s + 2} + \frac{9}{5} \frac{1}{s - 3}\right) = \frac{6}{5} \mathcal{L}^{-1}\left(\frac{1}{s + 2}\right) + \frac{9}{5} \mathcal{L}^{-1}\left(\frac{1}{s - 3}\right) = \frac{6}{5} e^{-2t} + \frac{9}{5} e^{3t}$$

You may use Wolfram Alpha to check your answer:

inverse laplace transform of $3s/(s^2-s-6)$

8. Use Partial fractions:

$$\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

Therefore,

$$8s^2 - 4s + 12 = A(s^2 + 4) + (Bs + C)s \Rightarrow 8s^2 - 4s + 12 = (A + B)s^2 + Cs + 4A$$

Therefore,

$$\begin{aligned} A + B &= 4 \\ C &= -4 \Rightarrow A = 3, C = -4, B = 1 \\ 4A &= 12 \end{aligned}$$

And we get:

$$\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = \frac{3}{s} + \frac{5s - 4}{s^2 + 4}$$

For the inverse Laplace transform, we look at this as:

$$3 \frac{1}{s} + 5 \frac{s}{s^2 + 4} - 2 \frac{2}{s^2 + 4}$$

so that our final answer is:

$$3 + 5 \cos(2t) - 2 \sin(2t)$$

In Wolfram Alpha, you can ask for just the partial fraction decomposition:

partial fraction (8s^2-4s+12)/(s(s^2+4))

10. Complete the square in the denominator:

$$\frac{2s - 3}{s^2 + 2s + 1 + 9} = \frac{2(s + 1) - 5}{(s + 1)^2 + 9} = 2 \frac{s + 1}{(s + 1)^2 + 3^2} - \frac{5}{3} \frac{3}{(s + 1)^2 + 3^2}$$

12. Take the Laplace transform of both sides:

$$s^2 Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = 0$$

Apply initial conditions and solve for $Y(s)$:

$$Y(s) = \frac{s + 3}{s^2 + 3s + 2} = \frac{2}{s + 1} + \frac{1}{s + 2}$$

so that

$$y(t) = 2e^{-t} - e^{-2t}$$

14. Same idea:

$$s^2 Y - sy(0) - y'(0) - 4(sY - y(0)) + 4Y = 0$$

Apply initial conditions and solve for $Y(s)$:

$$Y(s) = \frac{s - 3}{(s^2 - 4s + 4)} = \frac{(s - 3)}{(s - 2)^2} = \frac{(s - 2) - 1}{(s - 2)^2} = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}$$

Use table entries 2 and 11:

$$y(t) = e^{2t} - te^{2t}$$

20. Continuing in the same fashion as before, you should find that

$$Y(s) = \frac{s}{(s^2 + \omega^2)(s^2 + 4)} + \frac{s}{s^2 + \omega^2}$$

Use Partial Fractions on the first term:

$$\frac{s}{(s^2 + \omega^2)(s^2 + 4)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + 4}$$

so that (multiply it out and equate coefficients):

$$\begin{array}{l|l} s^3 & A + C = 0 \\ s^2 & B + D = 0 \\ s & 4A + C\omega^2 = 1 \\ \text{const} & 4B + D\omega^2 = 0 \end{array} \Rightarrow A = \frac{1}{4 - \omega^2}, C = -A, B = 0, D = 0$$

22. Same idea as before.

$$Y(s) = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}$$

Using partial fractions again, get that:

$$Y(s) = \frac{1/5}{s+1} - \frac{1}{5} \cdot \frac{s-3}{s^2-2s+2} + \frac{1}{s^2-2s+2}$$

Now complete the square in the denominator of the last two terms.

24. For the Laplace transform of the right hand side of the given differential equation, define $f(t)$ as the piecewise defined function. Then:

$$\int_0^{\infty} e^{-st} f(t) dt = \int_0^{\pi} e^{-st} dt + \int_{\pi}^{\infty} 0 dt = -\frac{e^{-st}}{s} \Big|_0^{\pi} = \frac{1 - e^{-\pi s}}{s}$$

Therefore, the Laplace transform will be:

$$Y(s) = \frac{1 - e^{-\pi s}}{s(s^2 + 4)} + \frac{s}{s^2 + 4}$$

28.

$$F'(s) = \frac{d}{ds} \left(\int_0^{\infty} e^{-st} f(t) dt \right) = \int_0^{\infty} \frac{d}{ds} (e^{-st}) f(t) dt = \int_0^{\infty} -te^{-st} f(t) dt = \mathcal{L}(-tf(t))$$

29. Use Exercise 28, which in this case is

$$-\mathcal{L}(-te^{at}) = -F'(s)$$

where $F(s) = \frac{1}{s-a}$. Therefore,

$$-\mathcal{L}(-te^{at}) = \frac{1}{(s-a)^2}$$

As in table entry 11 with $n = 1$.

30. Same idea as Exercise 29, except we differentiate twice (a little messy) to get:

$$\mathcal{L}(t^2 \sin(bt)) = \frac{d^2}{ds^2} \left(\frac{b}{s^2 + b^2} \right) = \frac{d}{ds} \left((-2bs)(s^2 + b^2)^{-2} \right) = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}$$

33. Differentiate table entry 9 with respect to s , then multiply by -1 .

37. This one is a bit tricky (A Challenge Problem!)

$$\mathcal{L}(g(t)) = \int_0^\infty e^{-st} \left[\int_0^t f(w) dw \right] dt = \int_0^\infty \int_0^t e^{-st} f(w) dw dt$$

In the (t, w) plane, we have $0 \leq w \leq t$ and $0 \leq t \leq \infty$. If we reverse the order of integration then $t \geq w$ and $0 \leq w \leq \infty$ (draw it in the (t, w) plane). Therefore,

$$\int_0^\infty f(w) \left[\int_w^\infty e^{-st} \right] dt dw = \int_0^\infty f(w) \frac{e^{-sw}}{s} dw$$

which is what we wanted: $\frac{1}{s} \mathcal{L}(f(t))$.