Quiz 6 SOLUTIONS:

1. Use complex exponentials to compute $\int e^{-t}\sin(2t)\,dt$.

   SOLUTION: We'll use $e^{(-1+2i)t}$ to do the integration. Then:

$$\int e^{(-1+2i)t}\,dt = \frac{e^{(-1+2i)t}}{-1+2i}$$

   We want the imaginary part of this, after the simplification (we can factor out the exponential):

$$e^{-t}\left(\frac{-1}{5}-\frac{2}{5}i\right)(\cos(2t)+i\sin(2t))$$

   The imaginary part will be:

$$e^{-t}\left(-\frac{2}{5}\cos(2t)-\frac{1}{5}\sin(2t)\right)$$

2. Use the definition of the Laplace transform and the complex exponential to compute the Laplace transform of $\sin(3t)$.

   SOLUTION: We'll be computing the integral of $e^{-st}\sin(3t)$ (compare with the previous problem). Then:

$$\int_0^\infty e^{-(s-3i)t}\,dt = -\frac{1}{s-3i}e^{-(s-3i)t}\Big|_{t=0}^{t\to\infty}$$

   You should give a quick reason to justify that, if $s>0$, then $e^{-st}e^{3it}$ will go to zero as $t\to\infty$ (it is because the size of $e^{-(s-3i)t}$ is the same as the size of e^{-st}, which goes to zero as $t\to\infty$). With this, we have:

$$\int_0^\infty e^{-(s-3i)t}\,dt = \frac{1}{s-3i}$$

   Rationalizing the denominator, we get:

$$\frac{s}{s^2+9}+i\,\frac{3}{s^2+9}$$

   so the Laplace transform of the $\sin(3t)$ is $\frac{3}{s^2+9}$.

3. Show (by finding K, a, M from the definition) that t^4 is of exponential order. You may use any Lemmas from class without proof (but you should be explicit about using them).

   SOLTUION:

$$t^4 = e^{\ln(t^4)} = e^{4\ln(t)} \le e^{4t}$$

   We used the Lemma from class that, for $t>1$, $t>\ln(t)$. Therefore, $K=1, a=4$ and $M=1$.

4. Complete the square to get an expression of the form: $k(s+a)^2+b$

$$2s^2+3s+4 = 2\left(s^2+\frac{3}{2}s+\quad\right)+4 = 2\left(s^2+\frac{3}{2}s+\frac{9}{16}\right)+4-2\frac{9}{16}$$

   Simplifying, we get:

$$2\left(s+\frac{3}{4}\right)^2+\frac{23}{8}$$

5. Show using integration by parts, that

$$\mathcal{L}(y''') = s^3Y(s)-s^2y(0)-sy'(0)-y''(0)$$

   SOLUTION: Integrate using a table:

$$\int_0^\infty e^{-st}y'''(t)\,dt \quad\Rightarrow\quad \begin{array}{c|r|r} + & e^{-st} & y'''(t) \\ - & -se^{-st} & y''(t) \\ + & s^2e^{-st} & y'(t) \\ - & -s^3e^{-st} & y(t) \end{array}$$

Therefore, the integral is (factoring out e^{-st}):

$$\mathrm{e}^{-st}\left(y''(t) + sy'(t) + s^2 y(t)\right|_{t=0}^{t\to\infty} + s^3 \int_0^\infty \mathrm{e}^{-st} y(t)\, dt$$

We assume that the Laplace transform of y, y', y'', y''' each exists, which implies that, as $t \to \infty$, each of the following terms goes to zero:

$$\mathrm{e}^{-st}y(t) \quad \mathrm{e}^{-st}y'(t) \quad \mathrm{e}^{-st}y''(t) \quad \mathrm{e}^{-st}y'''(t)$$

Therefore, the limit will be zero if $s > 0$. Finishing up, we get (re-ordering):

$$s^3 Y(s) - (s^2 y(0) + sy'(0) + y'(0))$$