2.1: Some Detailed Examples

Summary: Given $y' + p(t)y = f(t)$, we first find the integrating factor $\mu(t)$:

$$\mu(t) = \mathrm{e}^{\int p(t)\,dt}$$

Then multiply both sides of the DE by it:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)f(t)$$

Since $\mu' = \mu p$, then this becomes

$$\mu(t)y' + \mu'(t)y = \mu(t)f(t) \quad \Rightarrow \quad (\mu(t)y(t))' = \mu(t)f(t)$$

Then solve this by integrating both sides, then isolate y.

Example 1

$$ty' + (t+1)y = t \qquad y(\ln(2)) = 1$$

SOLUTION: First get the DE in standard form, $y' + p(t)y = f(t)$ by dividing by t:

$$y' + \frac{t+1}{t}y = 1$$

Now compute the integrating factor μ:

$$\mu(t) = \mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{\int 1+\frac{1}{t}\,dt} = \mathrm{e}^{t+\ln(t)} = \mathrm{e}^{t}\mathrm{e}^{\ln(t)} = t\mathrm{e}^{t}$$

Multiply both sides by the integrating factor so that the LHS becomes a "perfect derivative":

$$t\mathrm{e}^t(y' + (1+1/t)y) = t\mathrm{e}^t \quad \Rightarrow \quad \left(t\mathrm{e}^t\, y(t)\right)' = t\mathrm{e}^t$$

We integrate by parts using a table (see the Review sheet):

$$\int t\mathrm{e}^t\,dt \qquad \begin{array}{ccc} + & t & \mathrm{e}^t \\ - & 1 & \mathrm{e}^t \\ + & 0 & \mathrm{e}^t \end{array} \qquad = t\mathrm{e}^t - \mathrm{e}^t + C$$

so that

$$t\mathrm{e}^t y = t\mathrm{e}^t - \mathrm{e}^t + C$$

and

$$y(t) = 1 - \frac{1}{t} + \frac{C}{t}\mathrm{e}^{-t}$$

To solve for C, put in $t = \ln(2)$ and $y = 1$:

$$1 = 1 - \frac{1}{\ln(2)} + \frac{C}{\ln(2)}\mathrm{e}^{-\ln(2)} \quad \Rightarrow \quad 1 = \frac{C}{2} \quad \Rightarrow \quad C = 2$$

The solution is:

$$y(t) = 1 - \frac{1}{t} + \frac{2}{t}\mathrm{e}^{t}$$

Example 2 (2.1, #4)

Solve: $y' + (1/t)y = 3\cos(2t)$, $t > 0$

SOLUTION: The DE is already in standard form, so we can compute $\mu(t)$ directly:

$$\mu(t) = e^{\int 1/t\, dt} = e^{\ln(t)} = t$$

Multiply both sides of the DE by t so that the left side of the DE can be written as:

$$(y\,t)' = 3t\cos(2t)$$

Integrate the right side of the DE by parts using a table (see the Review sheet):

$$\begin{array}{c|c|c} + & t & 3\cos(2t) \\ - & 1 & \frac{3}{2}\sin(2t) \\ + & 0 & -\frac{3}{4}\cos(2t) \end{array} \qquad \Rightarrow \qquad \int 3t\cos(2t)\, dt = \frac{3}{2}t\sin(2t) + \frac{3}{4}\cos(2t)$$

Therefore,

$$y(t) = \frac{3}{2}\sin(2t) + \frac{3}{4t}\cos(2t) + \frac{C}{t}$$

NOTE: The following is incorrect:

$$y(t) = \frac{3}{2}\sin(2t) + \frac{3}{4t}\cos(2t) + C$$

Example 3 (2.1, # 31)

Solve the IVP below and describe how the initial value y_0 changes the nature of the solution $y(t)$.

$$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0$$

SOLUTION: The integrating factor can be computed quickly: $\mu(t) = e^{-\frac{3}{2}t}$ so that

$$\left(e^{-\frac{3}{2}t}y(t)\right)' = 3te^{-\frac{3}{2}t} + 2e^{-\frac{t}{2}}$$

The first term is integrated by parts (use a table), and the second is done directly. The general solution is then

$$y(t) = -2t - \frac{4}{3} - 4e^t + ce^{3t/2}$$

Putting in the initial value, we see that $c = \frac{16}{3} + y_0$. How does this change the nature of the solution? (Also see the direction field below)

- As t gets very large, and $c \neq 0$, the term $e^{3t/2}$ will dominate the expression.
- We see that if $y_0 > -16/3$, then $c > 0$ and so $y(t)$ will diverge to positive infinity.
- If $y_0 < -16/3$, then $c < 0$, and $y(t)$ will diverge (to negative infinity). We still go to negative infinity if $y_0 = -16/3$ as well.

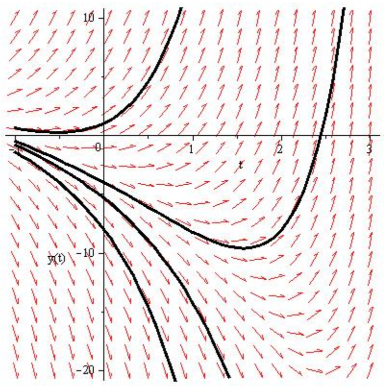

Figure 1: Direction field with some solution curves, Exercise 31, 2.1

Example 4 (2.1, #37)

Find a linear differential equation for which all solutions tend to $y = 4 - t^2$ as $t \to \infty$.

SOLUTION: From our expression in linear DE's, we might guess that:

$$y(t) = 4 - t^2 + Ce^{-t}$$

so that as $t \to \infty$, $y \to 4 - t^2$. Now we'll see if y satisfies a linear DE. We'll manipulate the expressions so that something of the form $y' + ay$ gets rid of the arbitrary constant C. One way to do it:

$$y' + y = (-2t - Ce^{-t}) + (4 - t^2 + Ce^{-t}) = 4 - 2t - t^2$$

Our ODE is therefore:

$$y' + y = 4 - 2t - t^2$$