## Sample Questions (Chapter 3, Math 244)

Exam Notes: You will not be allowed to have a calculator or any notes. You will have the formulas listed in the "Formula Page" on the summary.

1. State the Existence and Uniqueness theorem for linear, second order differential equations (non-homogeneous is the most general form):

2. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to

   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   (e) Consider the function:

   $$y(t) = \cos(t) - \sin(t)$$

   Then amplitude is 1, the period is 1 and the phase shift is 0.

   SOLUTION: False. For this question to make sense, we have to first write the function as $R\cos(\omega(t - \delta))$. In this case, the amplitude is $R$:

   $$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

   The period is $2\pi$ (the circular frequency, or natural frequency, is 1), and the phase shift $\delta$ is:

   $$\tan(\delta) = -1 \quad \Rightarrow \quad \delta = -\frac{\pi}{4}$$

3. Find values of $a$ for which **any** solution to:

   $$y'' + 10y' + ay = 0$$

   will tend to zero (that is, $\lim_{t\to 0} y(t) = 0$.

4. 
   - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

5. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

6. Find the solution to the initial value problem:

$$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

7. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

8. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

   - Start over, with Method of Undetermined Coefficients
   - Take the limit of your answer from Question 6 as $\omega \to \omega_0$.

9. For the following question, recall that the acceleration due to gravity is 32 ft/sec$^2$.

   An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1−lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight.

10. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

   Find a fundamental set of solutions using Abel's Theorem.

11. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

12. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

   (a) $y'' + 4y' + 4y = t^{-2} e^{-2t}$

   (b) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.

   (c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

   (d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

13. Rewrite the expression in the form $a + ib$: (i) $2^{i-1}$ (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

14. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) $-4$ (iv) $\sqrt{3} - i$

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:
$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):
$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of $a, b, c$.
If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:
$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

18. Solve the following Euler equations:
    (a) $t^2y'' + 2ty' - 2y = 0$
    (b) $t^2y'' + ty' + 9y = 0$

19. If $x = \ln(t)$ then verify that
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{1}{t}$$
and
$$\frac{d^2y}{dt^2} = \frac{d^2y/dx^2 - dy/dx}{t^2}$$
Hint on the second part:
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{1}{t}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\frac{1}{t}\right)\frac{dx}{dt}$$
We know that $dx/dt = 1/t$, so find an expression for the following using the product rule (think of $t$ as a function of $x$).
$$\frac{d}{dx}\left(\frac{dy}{dx}\frac{1}{t}\right)$$

20. Use Variation of Parameters to find a particular solution to the following, then verify your answer using the Method of Undetermined Coefficients:
$$4y'' - 4y' + y = 16e^{t/2}$$

21. Compute the Wronskian of two solutions of the given DE without solving it:
$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

22. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of $\alpha$ so that the solution tends to zero as $t \to \infty$.

23. Give the general solution to $y'' + y = \frac{1}{\sin(t)} + t$

24. A mass of 0.5 kg stretches a spring to 0.05 meters. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

25. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec$^2$). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

26. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

    (a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$
    (b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$
    (c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$
    (d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$
    (e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$