Power Series in Differential Equations

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  - Check the endpoints, $|x - x_0| = \rho$, separately to find the INTERVAL of CONVERGENCE.

The Ratio Test (to determine the radius of conv)

For a power series, the ratio test takes the following form:

$$\lim_{n\to\infty} \frac{|a_{n+1}||x-x_0|^n}{|a_n||x-x_0|^n} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x-x_0| = \left[\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\right]|x-x_0|$$

If the limit in the brackets is r, then overall the limit is:

$$\lim_{n\to\infty} \frac{|a_{n+1}||x-x_0|^n}{|a_n||x-x_0|^n} = r|x-x_0|$$

Conclusion: If $r|x-x_0| < 1$, the power series converges absolutely. Equivalently, the series converges absolutely if:

$$|x-x_0| < \frac{1}{r} = \rho$$

and this ρ is the RADIUS of CONVERGENCE.

Example 2

Find the radius and interval of convergence:

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Algebra before the limit to simplify:

$$\frac{|x+1|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x+1|^n} = \frac{n}{n+1} \cdot \frac{|x+1|}{3}$$

Now the limit:

$$\left(\lim_{n\to\infty} \frac{n}{n+1}\right) \cdot \frac{|x+1|}{3} = \frac{|x+1|}{3} < 1 \quad \Rightarrow \quad |x+1| < 3$$

Continuing the example

Find the *interval* of convergence:

$$|x+1| < 3 \quad \Rightarrow \quad -3 < x+1 < 3 \quad \Rightarrow \quad -4 < x < 2$$

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Checking endpoints:

- Substitute $x = -4$ into the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \Rightarrow \quad \text{Divergent}$$

- Substitute $x = 2$ into the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Radius of convergence: 3. Interval of convergence: $(-4, 2]$

The Taylor Series

Given a function f and a base point x_0, the Taylor series for f at x_0 is given by:

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

or

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

If $f(x)$ is equal to its Taylor series, then f is said to be *analytic*.

Definition: The Maclaurin series for f is the Taylor series based at $x_0 = 0$.

Example:

Find the Maclaurin series for $f(x) = \mathrm{e}^x$ at $x = 0$.

SOLUTION: Since $f^{(n)}(0) = 1$ for all n,

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

The radius of convergence is ∞.

Template Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Algebra on the Index, Example

Simplify to one sum that uses the term x^n:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=?}^{?} (\quad C_n \quad) x^n$$

SOLUTION: Try writing out the first few terms:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 2\cdot 3a_3 x + 3\cdot 4a_4 x^2 + 4\cdot 5a_5 x^3 + \cdots$$

$$\sum_{k=1}^{\infty} k a_k x^k = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

Powers of x don't line up: *To write this as a single sum, we need to manipulate the sums so that the powers of x line up.*

SOLUTION 1

Pad the second equation by starting at $k = 0$:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots$$

$$\sum_{k=0}^{\infty} ka_k x^k = 0 + a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

Substitute $n = m - 2$ (or $m = n + 2$) into the first sum, and $n = k$ into the second sum:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\sum_{k=0}^{\infty} ka_k x^k = \sum_{n=0}^{\infty} na_n x^n$$

Finishing the solution:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} ka_k x^{k-1}$$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n$$

$$= \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + na_n\right)x^n$$

Alternate Solution:

We could have started both indices using x^1 instead of x^0. Here are the sums again:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \left[3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots\right]$$

$$\sum_{k=1}^{\infty} k a_k x^k = \qquad a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$$

In this case,

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \sum_{m=3}^{\infty} m(m-1)a_m x^{m-2}$$

and let $n = m - 2$ to get

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n$$

Using Series in DEs

Given $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$ and $y'(x_0) = v_0$, assume y, p, q are analytic at x_0.

Ansatz:

$$y(t) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

so that

$$y'(t) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{and} \quad y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

The Big Picture

Substituting the series into the DE will give something like:

$$\sum(\ldots) + \sum(\ldots) + \sum(\ldots) = 0$$

We will want to write this in the form:

$$\sum(\quad C_n \quad)x^n = 0$$

Then we will set $C_n = 0$ for each n.