Homework (M 244): To Replace 7.1/7.2

1. Exercise 22 (Section 7.1, p 363)
2. Exercise 1, 4, 22, 23 (Section 7.2, p. 371-373)

For Exercises 3-8 below, define

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and I is the identity matrix:

3. Compute $(B - 4I)\boldsymbol{b}$
4. Compute $\det(B - 4I)$
5. Are AB and BA the same?
6. Compute $A^{-1}\boldsymbol{b}$
7. Verify: $A\boldsymbol{b} - 3\boldsymbol{b} = (A - 3I)\boldsymbol{b}$
8. Compute B^TB, $\text{Tr}(A)$, and $\text{Tr}(B)$
9. Short Answer:
   (a) Is every second order linear homogeneous differential equation (with constant coefficients) equivalent to a system of first order equations?
   (b) Can every 2×2 system of DEs be converted into an equivalent second order system? (Hint: To do our technique, what must be true?)
10. Give the solution to each system. If it has an infinite number of solutions, give your answer in vector form:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x - y &= 3 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 3 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= 2 \end{aligned}$$

11. Write each of the previous systems in matrix-vector form. Verify that the determinant of the first matrix is not zero, but is zero for the second and third.
12. Write each system of differential equations in matrix-vector form or write the system from the matrix-vector form:

$$\begin{aligned} x_1' &= 3x_1 - x_2 \\ x_2' &= 9x_1 - 3x_2 \end{aligned} \qquad \mathbf{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

13. Find the equilibrium solutions to the previous autonomous linear differential equations:
14. If $\mathbf{x}$ is as defined below, compute $\mathbf{x}'(t)$, and $\int_0^1 \mathbf{x}(t)\,dt$:

$$\mathbf{x}(t) = \begin{bmatrix} t^2 - 3 \\ 3e^t - 2e^{3t} \end{bmatrix}$$

15. If $\mathbf{y}(t) = A(t)\mathbf{c}$ is as defined below, compute $\mathbf{y}'(t)$, and $\int_0^1 \mathbf{y}(t)\,dt$. Are these the same as $A'(t)\mathbf{c}$ and $\int A(t)\,dt\,\mathbf{c}$?

$$\mathbf{y}(t) = \begin{bmatrix} t & 2t \\ 1 & \sin(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

16. Solve the system of equations given by first converting it into a second order linear ODE (then use Chapter 3 methods):

(a) $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$ (b) $\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}$

17. Convert the following second order differential equations into a system of autonomous, first order equations. Using methods from Chapter 3, give the solution to the system. An example follows before the exercises:

$$y'' + 3y' + 2y = 0$$

SOLUTION: We'll get the homogenous solution first. The roots to the characteristic equation are $-1, -2$. The general solution is:

$$y = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$$

To get an equivalent system, let $x_1 = y$ and $x_2 = y'$. Then

$$x_1' = y' = x_2 \qquad x_2' = y'' = -2y - 3y' = -2x_1 - 3x_2$$

so the system is (in matrix-vector form):

$$\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}$$

Since $x_1 = y$, then $x_1 = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$. Since $x_2 = y'$, then $x_2 = -C_1 \mathrm{e}^{-t} - 2C_2 \mathrm{e}^{-2t}$. In vector form, this means our solution is:

$$\mathbf{x} = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Here we go:

(a) $y'' + 4y' + 3y = 0$

(b) $y'' + 5y' = 0$

(c) $y'' + 4y = 0$

(d) $y'' - 2y' + y = 0$