Homework Solutions: Nonlinear systems

1. Fill in the following and under "Poincaré" classify the origin. Then, given the eigenvalues/eigenvectors, also write down the general solution to $\mathbf{x}' = A\mathbf{x}$. In the case that there is only one eigenvector, the second column of V shows the generalized eigenvector $\mathbf{w}$.

System	Tr(A)	det(A)	Δ	Poincare Fill in	λ	V
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$	2	5	-16	Spiral Source	$1+2i$	$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$	0	-1	(4)	Saddle	$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	0	4	-16	Center	$2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$	0	0	0	Uniform Motion	$0, 0$	$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}$

The solution to each system is given below:

System	Solution
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$	$C_1 e^t \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ -\cos(2t) + \sin(2t) \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$	$C_1 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$	$C_1 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$	$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left[t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right]$

2. Explain how the classification of the origin changes by changing the α in the system:

   (a) $\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$

   SOLUTION: The number line keeps track of where each quantity is zero, positive, negative: In this case, the trace is always -2, the $\det(A) = -\alpha$ (which is zero at 0), and the discriminant is

$\Delta = 4 + 4\alpha$ (which is zero at $\alpha = -1$). Now set up the number line for α:

$\det(A) = -\alpha$	$+$	$+$	$-$
$\Delta = 4 + 4\alpha$	$-$	$+$	$+$
	$\alpha < -1$	$-1 < \alpha < 0$	$\alpha > 0$
	Spiral Sink	Sink	Saddle

And at $\alpha = -1$, the origin is a degenerate sink, and at $\alpha = 0$, we have a line of stable fixed points.

(b) $\mathbf{x}' = \begin{bmatrix} 2 & \alpha \\ 1 & -1 \end{bmatrix} \mathbf{x}$

SOLUTION: Another number line with $Tr(A) = 1$ (which never changes sign), $\det(A) = -(2+\alpha)$ which is zero at -2, and $Delta = 9 + 4\alpha$, which is zero at $-9/4$.

$\det(A) = -(2 + \alpha)$	$+$	$+$	$-$
$\Delta = 9 + 4\alpha$	$-$	$+$	$+$
	$\alpha < -9/4$	$-9/4 < \alpha < -2$	$\alpha > 2$
	Spiral Source	Source	Saddle

And at $\alpha = -9/4$, the origin is a degenerate source, and at $\alpha = -2$, we have a line of unstable fixed points.

(c) $\mathbf{x}' = \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix} \mathbf{x}$

SOLUTION: In this case, the relevant numbers are:

$$\text{Tr}(A) = \alpha - 4 \qquad \det(A) = -4\alpha + 10 \qquad \Delta = \alpha^2 + 8\alpha - 24$$

Unfortunately, the roots to the quadratic aren't integers: $\Delta = 0$ for $alpha = -4 \pm 2\sqrt{10} \approx 2.32, -10.32$. We list the other zeros below and construct the number line:

Tr(A)	$-$	$-$	$-$	$-$	$+$
det(A)	$+$	$+$	$+$	$-$	$-$
Δ	$+$	$-$	$+$	$+$	$+$
	$\alpha < -10.3$	$-10.3 < \alpha < 2.3$	$2.3 < \alpha < 2.5$	$2.5 < \alpha < 4$	$\alpha > 4$
	Sink	Spiral Sink	Sink	Saddle	Saddle

The special cases are:

- $\alpha = -4 - 2\sqrt{10}$, Degenerate sink.
- $\alpha = -4 + 2\sqrt{10}$, Degenerate sink.
- $\alpha = 2.5$, Line of stable fixed points.
- $\alpha = 4$, Still a saddle

3/4 For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to t, as usual). Then linearize and classify them (We'll do the last two exercises together):

(a) $x' = x - xy$, $y' = y + 2xy$

SOLUTION: There are two equilibria, one at $(0, 0)$ and one at $(-1/2, 1)$. The Jacobian matrix is given below:, where it is also evaluated at the two equilibria (respectively):

$$J = \begin{bmatrix} 1-y & -x \\ 2y & 1+2x \end{bmatrix}$$

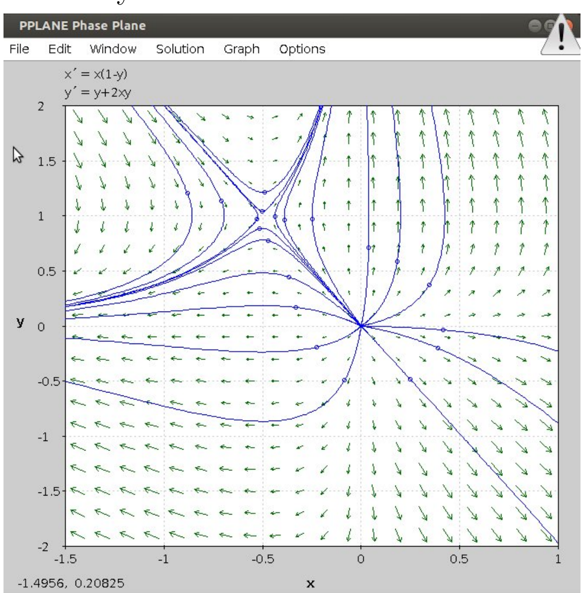
At the origin:

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \mathrm{Tr}(A) = 1 \\ \det(A) = 1 \\ \Delta = 0 \end{array} \Rightarrow \text{Degen Source}$$

Actually, this is the case we gave in class where we actually get two eigenvectors for $\lambda = 1, 1$. It is easy to solve this one directly.

$$J(-1/2,1) = \begin{bmatrix} 0 & 1/2 \\ 2 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \mathrm{Tr}(A) = 0 \\ \det(A) = -1 \end{array} \Rightarrow \text{Saddle}$$

And here is a screen shot of the direction field where we can verify our analysis:



(b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$

SOLUTION: From the first equation, $y = 0$ or $x + y = 2$. Putting that into the second equation, we get either $x = 0$ (so $(0, 0)$ is one solution), or $-2 - 2(2 - x) = 0$, or $x^2 - 2x - 1 = 0$, so that

$$x = 1 + \sqrt{2}, \quad y = 1 - \sqrt{2} \quad \text{and} \quad x = 1 - \sqrt{2} \quad y = 1 + \sqrt{2}$$

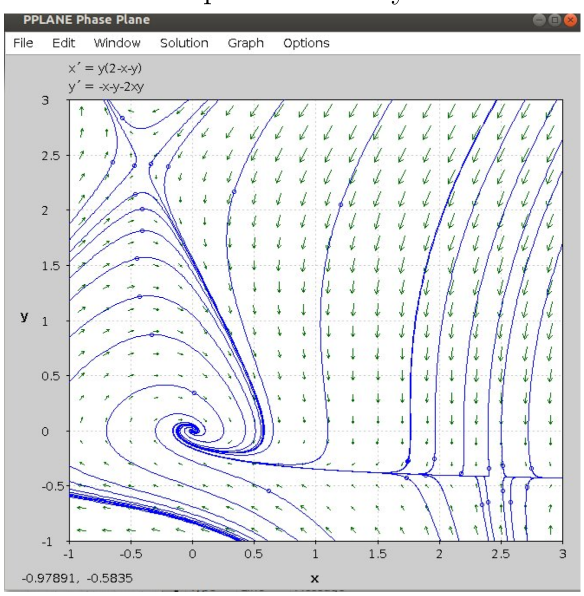
In this case, the Jacobian was:

$$J = \begin{bmatrix} -y & 2 - x - 2y \\ -1 - 2y & -1 - 2x \end{bmatrix}$$

Therefore, at the three equilibria, we had:

$$J(0,0) = \begin{bmatrix} 0 & 2 \\ -1 & -1 - 2x \end{bmatrix} \quad \Rightarrow \quad \begin{array}{r} \text{Tr}(A) = -1 \\ \det(A) = 2 \\ \Delta = -7 \end{array} \Rightarrow \text{Spiral Sink}$$

At the other two equilibria, we should find saddles (the numbers were a little messy). Here is the plot of the direction field, where we see these equilibria clearly:



(c) $x' = 1 + 2y$, $y' = 1 - 3x^2$

For the last one, there are two equilibria, $x = \pm 1/\sqrt{3}$ and $y = -1/2$. The Jacobian matrix is:

$$J = \begin{bmatrix} 0 & 2 \\ -6x & 0 \end{bmatrix}$$

At $x = 1/\sqrt{3}$, the equilibrium is a center, and at $x = -1/\sqrt{3}$, it is a saddle.