Summary- Elements of Chapters 7 and 9

We started with some basic matrix algebra- Be sure you know how to perform matrix-vector multiplication and matrix-matrix multiplication for 2×2 matrices.

Eigenvalues and Eigenvectors

1. Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that
$$A\mathbf{v} = \lambda\mathbf{v}$$
then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector.

2. Eigenvectors are not unique. That is, if $\mathbf{v}$ is an eigenvector for A, so is $k\mathbf{v}$ (prove it!).

3. If you're starting to compute them for the first time, start with the original definition and work through to the system:
$$A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow \begin{array}{ccc} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \Leftrightarrow \begin{array}{rrr} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \tag{1}$$
This system has a non-trivial solution for v_1, v_2 only if the determinant of coefficients is 0:
$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$
And this is the **characteristic equation**. We solve this for the eigenvalues:
$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \Leftrightarrow \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$
where $\mathrm{Tr}(A)$ is the trace of A (which we defined as $a+d$). For each λ, we must go back and solve Equation (1).

4. A note about notation: Often it is easier to use the notation $A - \lambda I$ to represent the matrix:
$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$
   - Using this notation, the characteristic equation becomes: $|A - \lambda I| = 0$.
   - Using this notation, the eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$
   - The generalized eigenvector $\mathbf{w}$ solves: $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Solve $\mathbf{x}' = A\mathbf{x}$

1. We make the ansatz:
$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda t} v_1 \\ \mathrm{e}^{\lambda t} v_2 \end{bmatrix}$$
which implies $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

2. To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \text{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \text{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

Computational Note: As in Chapter 3, our solutions here are real solutions- That means you should not have an i in your final answer.

- One eigenvalue, one eigenvector $\mathbf{v}$. Get $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$$

Computational Note: You should find that there are an infinite number of possible vectors $\mathbf{w}$- Just choose one convenient representative.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = \mathrm{e}^{rt}$	$\mathbf{x} = \mathrm{e}^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\det(A - \lambda I) = 0$
Real Solns	$y = C_1\mathrm{e}^{r_1 t} + C_2\mathrm{e}^{r_2 t}$	$\mathbf{x}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1\text{Re}(\mathrm{e}^{rt}) + C_2\text{Im}(\mathrm{e}^{rt})$	$\mathbf{x}(t) = C_1\text{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2\text{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$
SingleRoot	$y = \mathrm{e}^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(C_1\mathbf{v} + C_2\left(t\mathbf{v} + \mathbf{w}\right)\right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin (in Chapter 7) or other equilibrium solutions (in Chapter 9).

Solve General Nonlinear Equations

We don't have a method that will work on every system of nonlinear differential equations, although there are some tricks we can try with special cases- that is, given the system

$$\begin{array}{rl} \frac{dx}{dt} & = f(x,y) \\ \frac{dy}{dt} & = g(x,y) \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

And we might get lucky if it is in the form of an equation from Chapter 2.

Local Analysis of Nonlinear Equations

Most of our time was spent learning how to do a local linear analysis of systems of differential equations. That was, given the system:

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

- Find the equilibrium solutions ($f(x,y) = 0$ and $g(x,y) = 0$).
- At each equilibrium, we perform the local analysis by first linearizing, then we classify the equilibrium:
  - The linearization of the system at the point $x = a, y = b$ is given by:

    $$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

    where $u = x - a$, and $v = y - b$.

    NOTE 1: The solution of this local system is NOT the solution to the nonlinear system- Rather, we think of the solution to the nonlinear system as a "perturbation" of the solution to the local linear system.

    NOTE 2: The effect of the local linearization is to analyze the behavior of the full nonlinear system close to each of its equilibrium solutions.
  - Once we have the matrix, use the Poincaré Diagram to classify the equilibrium.