

# Quick Overview: Complex Numbers

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## 1 Initial Definitions

**Definition 1** *The complex number  $z$  is defined as:*

$$z = a + bi \tag{1}$$

where  $a, b$  are real numbers and  $i = \sqrt{-1}$ .

Remarks about the definition:

- Engineers typically use  $j$  instead of  $i$ .
- Examples of complex numbers:

$$5 + 2i, \quad 3 - \sqrt{2}i, \quad 3, \quad -5i$$

- Powers of  $i$ :

$$\begin{array}{ll} i^2 = -1 & i^3 = -i \\ i^4 = 1 & i^5 = i \\ i^6 = -1 & i^7 = -i \\ & \vdots \end{array}$$

- All real numbers are also complex (by taking  $b = 0$ ).

## 2 Visualizing Complex Numbers

A complex number is defined by its two real numbers. If we have  $z = a + bi$ , then:

**Definition 2** *The real part of  $a + bi$  is  $a$ ,*

$$\operatorname{Re}(z) = \operatorname{Re}(a + bi) = a$$

*The imaginary part of  $a + bi$  is  $b$ ,*

$$\operatorname{Im}(z) = \operatorname{Im}(a + bi) = b$$

To visualize a complex number, we use the complex plane  $\mathbb{C}$ , where the horizontal (or  $x$ -) axis is for the real part, and the vertical axis is for the imaginary part. That is,  $a + bi$  is plotted as the point  $(a, b)$ .

In Figure 1, we can see that it is also possible to represent the point  $a + bi$ , or  $(a, b)$  in polar form, by computing its modulus (or size), and angle (or argument):

$$r = |z| = \sqrt{a^2 + b^2} \quad \theta = \arg(z)$$

We have to be a bit careful defining  $\phi$ , since there are many ways to write  $\phi$  (and we could add multiples of  $2\pi$  as well). Typically, the argument of the complex number  $z = a + bi$  is defined to be the 4-quadrant “inverse tangent”<sup>1</sup> that returns  $-\pi < \theta \leq \pi$ .

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<sup>1</sup>For example, in Maple this special angle is computed as `arctan(b,a)`, and in Matlab the command is `atan2(b,a)`.

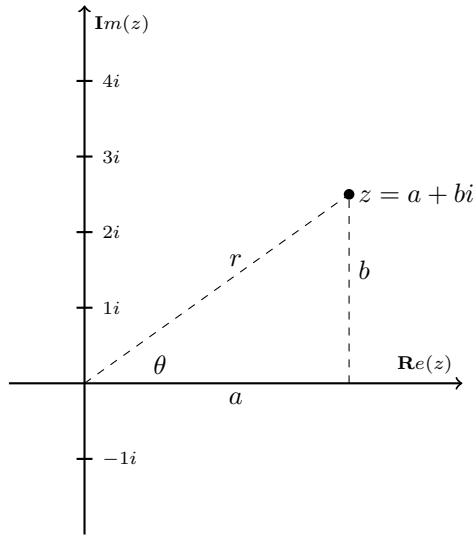


Figure 1: Visualizing  $z = a + bi$  in the complex plane. Shown are the modulus (or length)  $r$  and the argument (or angle)  $\theta$ .

That is, formally we can define the argument as:

$$\theta = \arg(a + bi) = \begin{cases} \tan^{-1}(b/a) & \text{if } a > 0 & \text{(Quad I and IV)} \\ \tan^{-1}(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0 & \text{(Quad II)} \\ \tan^{-1}(b/a) - \pi & \text{if } a < 0 \text{ and } b < 0 & \text{(Quad III)} \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 & \text{(Upper imag axis)} \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 & \text{(Lower imag axis)} \\ \text{Undefined} & \text{if } x = 0 \text{ and } y = 0 & \text{(The origin)} \end{cases}$$

## Examples

Find the modulus  $r$  and argument  $\theta$  for the following numbers (Hint: It is easiest to visualize these in the plane):

- $z = -3$ : SOLUTION:  $r = 3$  and  $\theta = \pi$
- $z = 2i$ : SOLUTION:  $r = 2$  and  $\theta = \pi/2$
- $z = -1 + i$ : SOLUTION:  $r = \sqrt{2}$  and  $\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$
- $z = -3 - 2i$  (Numerical approx from Calculator OK):  
SOLUTION:  $r = \sqrt{14}$  and  $\theta = \tan^{-1}(2/3) - \pi \approx 0.588 - \pi \approx -2.55$  rad

## 3 Operations on Complex Numbers

### 3.1 The Conjugate of a Complex Number

If  $z = a + bi$  is a complex number, then its *conjugate*, denoted by  $\bar{z}$  is  $a - bi$ . For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i \quad z = i \Rightarrow \bar{z} = -i \quad z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is its mirror image across the horizontal axis. If  $z$  has magnitude  $r$  and argument  $\theta$ , then  $\bar{z}$  has the same magnitude with a negative argument.

## 3.2 Addition/Subtraction, Multiplication/Division

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately:

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

To multiply, expand it as if you were multiplying polynomials:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

and simplify using  $i^2 = -1$ . Note what happens when you multiply a number by its conjugate:

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 = |z|^2$$

Division by complex numbers  $z, w$ :  $\frac{z}{w}$ , is defined by translating it to real number division (rationalize the denominator):

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Example:

$$\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{34} = \frac{-7}{34} + \frac{11}{34}i$$

## 4 The Polar Form of Complex Numbers

### 4.1 Euler's Formula

Any point on the unit circle can be written as  $(\cos(\theta), \sin(\theta))$ , which corresponds to the complex number  $\cos(\theta) + i\sin(\theta)$ . It is possible to show the following directly, but we'll use it as a definition:

**Definition (Euler's Formula):**  $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ .

### 4.2 Polar Form of $a + bi$ :

The polar form is defined as:

$$z = re^{i\theta} \quad \text{where} \quad r = |z| = \sqrt{a^2 + b^2} \quad \theta = \arg(z)$$

#### 4.2.1 Examples

Given the complex number in  $a + bi$  form, give the polar form, and vice-versa:

1.  $z = 2i$  SOLUTION: Since  $r = 2$  and  $\theta = \pi/2$ ,  $z = 2e^{i\pi/2}$

2.  $z = 2e^{-i\pi/3}$

We recall that  $\cos(\pi/3) = 1/2$  and  $\sin(\pi/3) = \sqrt{3}/2$ , so

$$z = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)) = 1 - \sqrt{3}i$$

## 5 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form:

$$\ln(a + bi) = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta$$

The logarithm of zero is left undefined (as in the real case). However, we can now compute the log of a negative number:

$$\ln(-1) = \ln(1 \cdot e^{i\pi}) = i\pi \quad \text{or the log of } i: \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

Note that the usual rules of exponentiation and logarithms still apply.

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like  $x^x$ ):

$$a^b = e^{b \ln(a)}$$

Example,  $2^i = e^{i \ln(2)} = \cos(\ln(2)) + i \sin(\ln(2))$

Example:  $\sqrt{1+i} = (1+i)^{1/2} = (\sqrt{2}e^{i\pi/4})^{1/2} = (2^{1/4})e^{i\pi/8}$

Example:  $i^i = e^{i \ln(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$

## 6 Real Polynomials and Complex Numbers

If  $ax^2 + bx + c = 0$ , then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of  $x^2 + 1 = 0$  are  $x = i$  and  $x = -i$ .

Check:

$$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some facts about polynomials when we allow complex roots:

1. An  $n^{\text{th}}$  degree polynomial can always be factored into  $n$  roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.
2. If  $a + bi$  is a root to a real polynomial, then  $a - bi$  must also be a root. This is sometimes referred to as “roots must come in conjugate pairs”.

## 7 Exercises

1. Suppose the roots to a cubic polynomial are  $a = 3$ ,  $b = 1 - 2i$  and  $c = 1 + 2i$ . Compute  $(x - a)(x - b)(x - c)$ .
2. Find the roots to  $x^2 - 2x + 10$ . Write them in polar form.
3. Show that:

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

4. For the following, let  $z_1 = -3 + 2i$ ,  $z_2 = -4i$ 
  - (a) Compute  $z_1 \bar{z}_2$ ,  $z_2/z_1$
  - (b) Write  $z_1$  and  $z_2$  in polar form.
5. In each problem, rewrite each of the following in the form  $a + bi$ :
  - (a)  $e^{1+2i}$
  - (b)  $e^{2-3i}$
  - (c)  $e^{i\pi}$
  - (d)  $2^{1-i}$
  - (e)  $e^{2-\frac{\pi}{2}i}$
  - (f)  $\pi^i$

6. For fun, compute the logarithm of each number:
  - (a)  $\ln(-3)$
  - (b)  $\ln(-1 + i)$
  - (c)  $\ln(2e^{3i})$