Exercise Set 3 Solutions

In this homework set, we will practice finding eigenvalues and eigenvectors when the eigenvalues are either complex or the matrix is defective.

1. Given a 2×2 defective matrix A with double eigenvalue λ, eigenvector $\mathbf{v}$ and generalized eigenvector $\mathbf{w}$, show that the function:

   $$e^{\lambda t}(t\mathbf{v} + \mathbf{w})$$

   solves the differential equation $\mathbf{x}' = A\mathbf{x}$.

   SOLUTION: Substitute it into the DE to see what needs to be true. First, we compute $\mathbf{x}'$, then we'll compare that to $A\mathbf{x}$. To differentiate, we need to use the product rule:

   $$\mathbf{x}' = \lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}) + e^{\lambda t}\mathbf{v}$$

   We'll multiply $A\mathbf{x}$ and use the fact that $A\mathbf{v} = \lambda\mathbf{v}$:

   $$Ae^{\lambda t}(t\mathbf{v} + \mathbf{w}) = e^{\lambda t}(tA\mathbf{v} + A\mathbf{w}) = e^{\lambda t}(t\lambda v + A\mathbf{w})$$

   Set them equal:

   $$\lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}) + e^{\lambda t}\mathbf{v} = e^{\lambda t}(t\lambda\mathbf{v} + A\mathbf{w})$$

   Divide by $e^{\lambda t}$ and simplify:

   $$\lambda\mathbf{w} + \mathbf{v} = A\mathbf{w} \quad \Rightarrow \quad (A - \lambda I)\mathbf{w} = \mathbf{v}$$

   You may leave this equation as $A\mathbf{w} - \lambda\mathbf{w} = \mathbf{v}$ as well. This is exactly the condition that we defined $\mathbf{w}$ to have.

2. Exercises 1, 3, pg. 409 (Section 7.6, solve with complex evals/evecs)

   Solutions in the text.

3. Exercises 1, 3, 7, pg. 429 (Section 7.8, solve with degenerate matrix)

   Solutions in the text.

4. Exercises 13, 15, pg 410 (Section 7.6, try to analyze with parameter- We'll do these more in depth later as well).

   Solutions in the text.

5. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, spiral sink, spiral source, saddle, or none of the above.

(a) $\lambda = -1 + 2i \qquad \mathbf{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

SOLUTION: We can classify without looking at the solution. The eigenvalues are complex and not pure imaginary. The real part is -1, which means that e^{-t} will be a factor of the solution. Therefore, we have a *spiral sink*.

The general solution is found by first computing $e^{\lambda t}\mathbf{v}$:

$$e^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1-i \\ 2 \end{bmatrix} =$$

$$e^{-t} \begin{bmatrix} (\cos(2t) + \sin(2t)) + i(\sin(2t) - \cos(2t)) \\ 2\cos(2t) + i \cdot 2\sin(2t) \end{bmatrix}$$

Then we take $C_1 \mathrm{Re}(e^{\lambda t}\mathbf{v}) + C_2 \mathrm{Im}(e^{\lambda t}\mathbf{v})$:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} \cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin(2t) - \cos(2t) \\ 2\sin(2t) \end{bmatrix}$$

(b) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

SOLUTION: With two real, distinct solutions, the origin will be either a sink, a source or a saddle. Since the eigenvalues are opposite in sign, the origin is a saddle. The solution is:

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

SOLUTION: This is a double root- The origin is still a sink, since all nearby solutions will tend to zero (both roots are negative). Furthermore, the general solution is:

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + +C_2 e^{-2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$$

(d) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

SOLUTION: This is almost identical to part (b), and the origin is a saddle.

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(e) $\lambda = 1 + 3i \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$

SOLUTION: Similar to part (a), but in this case (because the real part of λ is $+1$, the origin will be a spiral source instead of a sink.

$$\mathbf{x}(t) = C_1 \mathrm{e}^t \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + C_2 \mathrm{e}^{-t} \begin{bmatrix} \sin(3t) - \cos(2t) \\ \sin(3t) - \cos(3t) \end{bmatrix}$$

(f) $\lambda = 2i \qquad \mathbf{v} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$

SOLUTION: In the case of a pure imaginary λ, the solutions will be periodic (closed curves in the phase plane). In this case, the origin is called a "center" (we'll discuss this more next time). The solution proceeds as usual by first computing $\mathrm{e}^{\lambda t}\mathbf{v}$, etc:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) + \cos(2t) \\ \sin(2t) \end{bmatrix}$$

6. Use the Poincaré Diagram on page 497, where $p = \mathrm{Tr}(A)$ and $q = \det(A)$ to determine the stability of the origin for $\mathbf{x}' = A\mathbf{x}$, if A is given below:

NOTE FOR THE SOLUTION: The curve shown is where the discriminant is zero:

$$\Delta = 0 \qquad \text{or} \qquad (\mathrm{Tr}(A))^2 - 4\det(A) = 0$$

If we think of $\mathrm{Tr}(A)$ as an $x-$variable and $\det(A)$ as a $y-$variable, then this becomes

$$x^2 - 4y = 0 \quad \Rightarrow \quad y = \frac{1}{4}x^2$$

And this is the parabola you see. Inside the parabola is where the discriminant is negative, outside is where the discriminant is positive. With the three numbers $\mathrm{Tr}(A)$, $\det(A)$ and Δ, you should be able to find where you are in the diagram. I've attached the solutions to the following by scanning in a hand drawing.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$

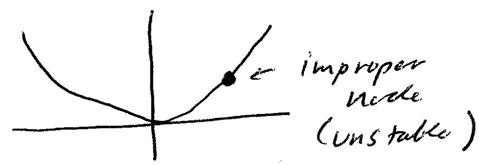
(c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

6(a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow$ $Tr(A) = 4 > 0$

$det(A) = 4 > 0$

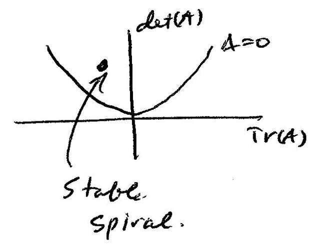
$\Delta = 4^2 - 4 \cdot 4 = 0$



6 (b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \Rightarrow$ $Tr(A) = -1 < 0$

$det(A) = 5/4 > 0$

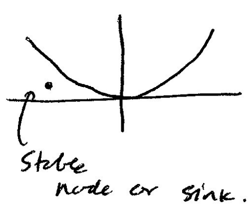
$\Delta = 1 - 4 \cdot 5/4 < 0$



6(c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix} \Rightarrow$ $Tr(A) = -5/4 < 0$

$det(A) = 1/4 > 0$

$\Delta = \frac{25}{16} - 4 \cdot \frac{1}{4} > 0$



6 (d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \Rightarrow$ $Tr(A) = 2$

$det(A) = 5$

$\Delta = 4 - 4 \cdot 5 < 0$