

## Solutions to Review Questions: Exam 3

1. What is the ansatz we use for  $y$  in

- Chapter 6? SOLUTION:  $y(t)$  is piecewise continuous and is of exponential order (so that  $Y(s)$  exists).
- Section 5.2? SOLUTION:  $y(x)$  is analytic at  $x = x_0$ . That is,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- Section 5.3-5.4: Not on exam.

2. Finish the definitions:

- The Heaviside function,  $u_c(t)$ :

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \quad c > 0$$

- The Dirac  $\delta$ -function:  $\delta(t - c)$

$$\delta(t - c) = \lim_{\tau \rightarrow 0} d_{\tau}(t - c)$$

where

$$d_{\tau}(t - c) = \begin{cases} \frac{1}{2\tau} & \text{if } c - \tau < t < c + \tau \\ 0 & \text{elsewhere} \end{cases}$$

- Define the convolution:  $(f * g)(t)$

$$(f * g)(t) = \int_0^t f(t - u)g(u) du$$

- A function is of **exponential order** if:  
there are constants  $M, k$ , and  $a$  so that

$$|f(t)| \leq Me^{kt} \quad \text{for all } t \geq a$$

3. Use the definition of the Laplace transform to determine  $\mathcal{L}(f)$ :

(a)

$$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

$$\int_0^{\infty} e^{-st} f(t) dt = \int_0^2 3e^{-st} dt + \int_2^{\infty} (6 - t)e^{-st} dt$$

The second antiderivative is found by integration by parts:

$$\int_2^{\infty} (6 - t)e^{-st} dt \Rightarrow \begin{array}{ccc} + & 6 - t & e^{-st} \\ - & -1 & (-1/s)e^{-st} \\ + & 0 & (1/s^2)e^{-st} \end{array} \Rightarrow e^{-st} \left( -\frac{6 - t}{s} + \frac{1}{s^2} \right) \Big|_2^{\infty}$$

Putting it all together,

$$-\frac{3}{s}e^{-st}\Big|_0^2 + \left(0 - e^{-2s}\left(-\frac{4}{s} + \frac{1}{s^2}\right)\right) = -\frac{3e^{-2s}}{s} + \frac{3}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = \frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

NOTE: Did you remember to *antidifferentiate* in the third column?

(b)

$$f(t) = \begin{cases} e^{-t}, & 0 \leq t < 5 \\ -1, & t \geq 5 \end{cases}$$

$$\int_0^\infty e^{-st} f(t) dt = \int_0^5 e^{-st} e^{-t} dt + \int_5^\infty -e^{-st} dt = \int_0^5 e^{-(s+1)t} dt + \int_5^\infty -e^{-st} dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}e^{-(s+1)t}\Big|_0^5 + \frac{1}{s}e^{-st}\Big|_5^\infty = \frac{1}{s+1} - \frac{e^{-5(s+1)}}{s+1} + 0 - \frac{e^{-5s}}{s}$$

4. Check your answers to Problem 3 by rewriting  $f(t)$  using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

(a)  $f(t) = 3(u_0(t) - u_2(t)) + (6-t)u_2(t) = 3 - 3u_2(t) + (6-t)u_2(t) = 3 + (3-t)u_2(t)$

For the second term, notice that the table entry is for  $u_c(t)h(t-c)$ . Therefore, if

$$h(t-2) = 3-t \quad \text{then} \quad h(t) = 3-(t+2) = 1-t \quad \text{and} \quad H(s) = \frac{1}{s} - \frac{1}{s^2}$$

Therefore, the overall transform is:

$$\frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

(b)  $f(t) = e^{-t}(u_0(t) - u_5(t)) - u_5(t)$

We can rewrite  $f$  in preparation for the transform:

$$f(t) = e^{-t}u_0(t) - e^{-t}u_5(t) - u_5(t)$$

For the middle term,

$$h(t-5) = e^{-t} \Rightarrow h(t) = e^{-(t+5)} = e^{-5}e^{-t}$$

so the overall transform is:

$$F(s) = \frac{1}{s+1} - e^{-5}\frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s}$$

5. Write the following functions in piecewise form (thus removing the Heaviside function):

(a)  $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$

SOLUTION: First, notice that  $(t+2)$  is turned “on” at time 2. At time  $t = 3$ ,  $\sin(t)$  joins the first function, and at time  $t = 4$ , we subtract the function  $t+2$  back off.

$$\begin{cases} 0 & \text{if } 0 \leq t < 2 \\ t+2 & \text{if } 2 \leq t < 3 \\ t+2+\sin(t) & \text{if } 3 \leq t < 4 \\ \sin(t) & \text{if } t \geq 4 \end{cases}$$

(b)  $\sum_{n=1}^4 u_{n\pi}(t) \sin(t - n\pi)$

SOLUTION: First (you can determine this graphically)  $\sin(t - \pi) = -\sin(t)$ , and  $\sin(t - 2\pi) = \sin(t)$ , and  $\sin(t - 3\pi) = -\sin(t)$ , etc.- You should simplify these. Therefore:

$$\begin{cases} 0 & \text{if } 0 \leq t < \pi \\ \sin(t - \pi) & \text{if } \pi \leq t < 2\pi \\ \sin(t - \pi) + \sin(t - 2\pi) & \text{if } 2\pi < t < 3\pi \\ \sin(t - \pi) + \sin(t - 2\pi) + \sin(t - 3\pi) & \text{if } 3\pi \leq t < 4\pi \\ \sin(t - \pi) + \sin(t - 2\pi) + \sin(t - 3\pi) + \sin(t - 4\pi) & \text{if } t \geq 4\pi \end{cases} =$$

$$\begin{cases} 0 & \text{if } 0 \leq t < \pi \\ -\sin(t) & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } 2\pi < t < 3\pi \\ -\sin(t) & \text{if } 3\pi \leq t < 4\pi \\ 0 & \text{if } t \geq 4\pi \end{cases}$$

6. Determine the Laplace transform:

(a)  $t^2 e^{-9t}$

$$\frac{2}{(s+9)^3}$$

(b)  $e^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

(c)  $t^2 y'(t)$ . Use Table Entry 16,  $\mathcal{L}(-t^n f(t)) = F^{(n)}(s)$ . In this case,  $F(s) = sY(s) - y(0)$ , so  $F'(s) = sY'(s) + Y(s)$  and  $F''(s) = sY''(s) + 2Y'(s)$ .

(d)  $e^{3t} \sin(4t)$

$$\frac{4}{(s-3)^2 + 16}$$

(e)  $e^t \delta(t-3)$

In this case, notice that  $f(t)\delta(t-c)$  is the same as  $f(c)\delta(t-c)$ , since the delta function is zero everywhere except at  $t=c$ . Therefore,

$$\mathcal{L}(e^t \delta(t-c)) = e^3 e^{-3s}$$

(f)  $t^2 u_4(t)$

In this case, let  $h(t-4) = t^2$ , so that

$$h(t) = (t+4)^2 = t^2 + 8t + 16 \Rightarrow H(s) = \frac{2 + 8s + 16s^2}{s^3}$$

and the overall transform is  $e^{-4s}H(s)$ .

7. Find the inverse Laplace transform:

(a)  $\frac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6} = \frac{2s-1}{(s^2-4s+4)+2} = 2 \frac{s-1/2}{(s-2)^2+2} =$$

In the numerator, make  $s - \frac{1}{2}$  into  $s - 2 + \frac{3}{2}$ , then

$$2 \left( \frac{s-2}{(s-2)^2+2} + \frac{3}{2\sqrt{2}} \frac{\sqrt{2}}{(s-2)^2+2} \right) \Rightarrow 2e^{2t} \cos(\sqrt{2}t) + \frac{3}{\sqrt{2}} e^{2t} \sin(\sqrt{2}t)$$

(b)  $\frac{7}{(s+3)^3} = \frac{7}{2!} \frac{2!}{(s+3)^3} \Rightarrow \frac{7}{2} t^2 e^{-3t}$

(c)  $\frac{e^{-2s}(4s+2)}{(s-1)(s+2)} = e^{-2s}H(s)$ , where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{2}{s+2} \Rightarrow h(t) = 2e^t + 2e^{-2t}$$

and the overall inverse:  $u_2(t)h(t-2)$ .

(d)  $\frac{3s-1}{2s^2-8s+14}$  Complete the square in the denominator, factoring the constants out:

$$\frac{3s-1}{2(s^2-8s+5)} = \frac{3}{2} \cdot \frac{s-1/3}{(s-2)^2+3} = \frac{3}{2} \left( \frac{s-2}{(s-2)^2+3} + \frac{5}{3} \cdot \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s-2)^2+3} \right)$$

The inverse transform is:

$$\frac{3}{2} e^{2t} \cos(\sqrt{3}t) + \frac{5}{2\sqrt{3}} e^{2t} \sin(\sqrt{3}t)$$

(e)  $(e^{-2s} - e^{-3s}) \frac{1}{s^2+s-6} = (e^{-2s} - e^{-3s}) H(s)$

Where:

$$H(s) = \frac{1}{s^2+s-6} = \frac{1}{5} \frac{1}{s-2} - \frac{1}{5} \frac{1}{s+3}$$

so that

$$h(t) = \frac{1}{5} e^{2t} - \frac{1}{5} e^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2) - u_3(t)h(t-3)$$

8. For the following differential equations, solve for  $Y(s)$  (the Laplace transform of the solution,  $y(t)$ ). Do not invert the transform.

(a)  $y'' + 2y' + 2y = t^2 + 4t$ ,  $y(0) = 0$ ,  $y'(0) = -1$

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

so that

$$Y(s) = \frac{2}{s^3(s^2 + 2s + 2)} + \frac{4}{s^2(s^2 + 2s + 2)} - \frac{1}{s^2 + 2s + 2}$$

(b)  $y'' + 9y = 10e^{2t}$ ,  $y(0) = -1$ ,  $y'(0) = 5$

$$s^2Y + s - 5 + 9Y = \frac{10}{s - 2} \Rightarrow Y(s) = \frac{10}{(s - 2)(s^2 + 9)} - \frac{s - 5}{s^2 + 9}$$

(c)  $y'' - 4y' + 4y = t^2e^t$ ,  $y(0) = 0$ ,  $y'(0) = 0$

$$(s^2 - 4s + 4)Y = \frac{2}{(s - 1)^3} \Rightarrow Y(s) = \frac{2}{(s - 1)^3(s - 2)^2}$$

9. Solve the given initial value problems using Laplace transforms:

(a)  $2y'' + y' + 2y = \delta(t - 5)$ , zero initial conditions.

$$Y = \frac{e^{-5s}}{2s^2 + s + 2} = e^{-5s}H(s)$$

where

$$H(s) = \frac{1}{2s^2 + s + 2} = \frac{1}{2} \frac{1}{s^2 + \frac{1}{2}s + 1} = \frac{1}{2} \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} = \frac{1}{2} \frac{4}{\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

Therefore,

$$h(t) = \frac{2}{\sqrt{15}} e^{-1/4t} \sin\left(\frac{\sqrt{15}}{4}t\right)$$

And the overall solution is  $u_5(t)h(t - 5)$

(b)  $y'' + 6y' + 9y = 0$ ,  $y(0) = -3$ ,  $y'(0) = 10$

$$s^2Y + 3s - 10 + 6(sY + 3) + 9Y = 0 \Rightarrow Y = -\frac{3s + 8}{(s + 3)^2}$$

Partial Fractions:

$$-\frac{3s + 8}{(s + 3)^2} = -\frac{3}{(s + 3)} + \frac{1}{(s + 3)^2} \Rightarrow y(t) = -3e^{-3t} + te^{-3t}$$

(c)  $y'' - 2y' - 3y = u_1(t)$ ,  $y(0) = 0$ ,  $y'(0) = -1$

$$Y = e^{-s} \frac{1}{s(s-3)(s+1)} + \frac{1}{(s+1)(s-3)} = e^{-s} H(s) + \frac{1}{4} \frac{1}{s-3} - \frac{1}{4} \frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3} \frac{1}{s} + \frac{1}{12} \frac{1}{s-3} + \frac{1}{4} \frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t}$$

and the overall solution is:

$$y(t) = \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} + u_1(t)h(t-1)$$

(d)  $y'' + 4y = \delta(t - \frac{\pi}{2})$ ,  $y(0) = 0$ ,  $y'(0) = 1$

$$Y = e^{-\pi/2 s} \frac{1}{s^2 + 4} + \frac{1}{s^2 + 4}$$

Therefore,

$$y(t) = \frac{1}{2} \sin(2t) + u_{\pi/2}(t) \frac{1}{2} \sin(2(t - \pi/2))$$

(e)  $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi)$ ,  $y(0) = y'(0) = 0$ . Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty} e^{-2k\pi s} \frac{1}{s^2 + 1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty} u_{2k\pi}(t) \sin(t - 2\pi k) = \sum_{k=1}^{\infty} u_{2\pi k}(t) \sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2\pi \\ \sin(t) & \text{if } 2\pi \leq t < 4\pi \\ 2\sin(t) & \text{if } 4\pi \leq t < 6\pi \\ 3\sin(t) & \text{if } 6\pi \leq t < 8\pi \\ \vdots & \vdots \end{cases}$$

10. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

(a)  $4y'' + 4y' + 17y = g(t)$   $y(0) = 0, y'(0) = 0$

SOLUTION: First, note that

$$4s^2 + 4s + 17 = 4(s^2 + s + 17/4) = 4((s + 1/2)^2 + 4)$$

Therefore,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + 17} = G(s) \cdot \frac{1}{8} \frac{2}{(s + \frac{1}{2})^2 + 2^2}$$

Therefore,

$$y(t) = g(t) * \frac{1}{8} e^{-t/2} \sin(2t)$$

- (b)  $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$ , with  $y(0) = 1$  and  $y'(0) = -1$ .

SOLUTION: Take the Laplace transform of both sides:

$$(s^2Y - s + 1) + (sY - 1) \frac{5}{4}Y = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

so that

$$Y(s) = \frac{1 - e^{-\pi s}}{s(s^2 + s + 5/4)} + \frac{s}{s^2 + s + 5/4}$$

For the second term,

$$\frac{s}{s^2 + s + 5/4} = \frac{s}{(s + \frac{1}{2})^2 + 1} = \frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + 1} - \frac{1}{2} \frac{1}{(s + \frac{1}{2})^2 + 1}$$

For the first term, treat it like:

$$(e^{-0s} - e^{-\pi s}) H(s)$$

where

$$H(s) = \frac{1}{s} \cdot \frac{1}{s^2 + s + \frac{5}{4}} = \frac{1}{s} \cdot \frac{1}{(s + \frac{1}{2})^2 + 1}$$

so that

$$h(t) = 1 * e^{-t/2} \sin(t)$$

Therefore, the overall answer is:

$$y(t) = h(t) - u_\pi(t)h(t - \pi) + e^{-t/2} \left( \cos(t) - \frac{1}{2} \sin(t) \right)$$

11. Short Answer:

- (a)  $\int_0^\infty \sin(3t) \delta(t - \frac{\pi}{2}) dt = \sin(3\pi/2) = -1$ , since

$$\int_0^\infty f(t) \delta(t - c) dt = f(c)$$

- (b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t - c), \quad y(0) = 0$$

SOLUTION:

$$sY = e^{-cs} \Rightarrow Y = \frac{e^{-cs}}{s}$$

so that  $y(t) = u_c(t)$ . Therefore, the “derivative” of the Heaviside function is the Dirac  $\delta$ -function!

- (c) Use Laplace transforms to solve for  $F(s)$ , if

$$f(t) + 2 \int_0^t \cos(t-x)f(x) dx = e^{-t}$$

(So only solve for the transform of  $f(t)$ , don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2+1} = \frac{1}{s+1} \Rightarrow F(s) \left( \frac{(s+1)^2}{s^2+1} \right) = \frac{1}{s+1}$$

so that

$$F(s) = \frac{s^2+1}{(s+1)^3}$$

- (d) In order for the Laplace transform of  $f$  to exist,  $f$  must be?  
 *$f$  must be piecewise continuous and of exponential order*
- (e) Can we assume that the solution to:  $y'' + p(x)y' + q(x)y = u_3(x)$  is a power series?  
 SOLUTION: No. Notice that the second derivative is not continuous at  $x = 3$ , but the second derivative of the power series would be.
- (f) Use the table to find the Laplace transform of  $e^{-2t} \sinh(\sqrt{3}t)$ .  
 SOLUTION: Use Table Entries 14 and 7:

$$\mathcal{L}(e^{-2t} \sinh(\sqrt{3}t)) = F(s+2)$$

where  $F(s)$  is the Laplace transform of  $\sinh(\sqrt{3}t)$ :

$$F(s) = \frac{\sqrt{3}}{s^2-9} \Rightarrow \text{Overall Answer: } F(s+2) = \frac{\sqrt{3}}{(s+2)^2-9}$$

12. Let  $f(t) = t$  and  $g(t) = u_2(t)$ .

- (a) Use the Laplace transform to compute  $f * g$ .

To use the table,

$$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{e^{-2s}}{s} = e^{-2s} \frac{1}{s^3} = e^{-2s} H(s)$$

so that  $h(t) = \frac{1}{2}t^2$ . The inverse transform is then

$$u_2(t) \frac{1}{2}(t-2)^2$$

- (b) Verify your answer by directly computing the integral.

By direct computation, we'll choose to "flip and shift" the function  $t$ :

$$f * g = \int_0^t (t-x)u_2(x) dx$$

Notice that  $u_2(x)$  is zero until  $x = 2$ , then  $u_2(x) = 1$ . Therefore, if  $t \leq 2$ , the integral is zero. If  $t \geq 2$ , then:

$$\int_0^t (t-x)u_2(x) dx = \int_2^t t-x dx = tx - \frac{1}{2}x^2 \Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$



valid for  $t \geq 2$ , zero before that. This means that the convolution is:

$$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

13. If  $a_0 = 1$ , determine the coefficients  $a_n$  so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by  $\sum_{n=0}^{\infty} a_n x^n$ .

SOLUTION: The recognition problem is a little difficult, but we should be able to get the coefficients:

$$a_{k+2} = -\frac{2}{k+1} a_k$$

Just doing the straight computations, we get:

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \dots$$

To see the pattern, it is easiest to look at the general terms (Typically, I wouldn't ask for the recognition part on the exam, but you should be able to get the first few computations, as we did above):

$$\begin{array}{lll} a_1 & = -2a_0 & = \frac{(-2)}{1!} a_0 \\ a_2 & = -\frac{2}{2} a_0 = 2a_0 & = \frac{4}{2!} a_0 \\ a_3 & = -\frac{2}{3} a_2 = -\frac{4}{3} a_0 & = \frac{-8}{3!} a_0 \\ \vdots & & \vdots \end{array}$$

The series is for  $e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

14. Write the following as a single sum in the form  $\sum_{k=2}^{\infty} c_k (x-1)^k$  (with a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1) a_n (x-1)^{n-2} + x(x-2) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

In front of the second sum we have  $x^2 - 2x$ , but we can't bring that directly into the sum since we have powers of  $(x-1)$ . But, we might recognize that:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x-1)^2 - 1$$

Therefore, the second sum can be expanded into two sums:

$$\begin{aligned} ((x-1)^2 - 1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} &= (x-1)^2 \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \\ &= \sum_{n=1}^{\infty} n a_n (x-1)^{n+1} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \end{aligned}$$

Now we have three sums to work with

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In the first sum, the first non-zero term has  $(x-1)^0$ , the second sum begins with  $(x-1)^2$ , and the last sum starts with  $(x-1)^0$ . We could shift the second index to start at  $n=0$ , but then the sum begins with  $(x-1)^1$ . We'll have to break off the constant terms from the first two sums:

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1)(x-1)^{n-2}$$

and similarly

$$- \sum_{n=1}^{\infty} na_n(x-1)^{n-1} = -a_1 - \sum_{n=2}^{\infty} na_n(x-1)^{n-1}$$

Now we can bring all three sums together. In the first sum, we'll substitute  $k = n - 2$  (or  $n = k + 2$ ). In the middle sum,  $k = n + 1$  (or  $n = k - 1$ ), and in the third sum,  $k = n - 1$  (or  $n = k + 1$ ). With these substitutions, we get:

$$2a_2 - a_0 + \sum_{k=1}^{\infty} ((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} - (k+1)a_{k+1})(x-1)^k$$

NOTE: The question asked for the index to start at  $k = 2$  instead of  $k = 1$ - It's OK to do it either way; mainly, I wanted to see you put the sums together as one.

15. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \quad y(0) = 1, y'(0) = 1$$

SOLUTION: The idea behind this question is to get you to think about the kinds of solutions we get from the Laplace transform. If we do not require  $y$  to be continuous, then this DE is actually two differential equations:

$$y'' + 4y = 0, \quad y(0) = 1, y'(0) = 1 \quad \text{valid for } t \leq 1$$

And

$$y'' + 4y = 1 \quad y(1), y'(1) \text{ arbitrary}, \text{ valid for } t > 1$$

The general solution is then:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \leq 1 \\ c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

If we require  $y(t)$  to be continuous (a very common assumption), then we get the answer that comes from using Laplace transforms. Writing the answer in piecewise form:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \leq 1 \\ -\frac{1}{4}\cos(2(t-1)) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

16. Use the table to find an expression for  $\mathcal{L}(ty')$ . Use this to convert the following DE into a linear first order DE in  $Y(s)$  (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

SOLUTION: For the first part, use Table Entry 19. In particular,

$$\mathcal{L}(tf(t)) = -F'(s)$$

where, in our case,  $f(t) = y'(t)$ , so that  $F(s) = sY - y(0)$ . Therefore,

$$\mathcal{L}(tf(t)) = -(Y - sY') = sY' - Y$$

Substituting this into the DE, we get:

$$Y' + \left( \frac{s^2 - 3s - 6}{3s} \right) Y = \frac{1}{s}$$

17. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a)  $2y'' + xy' + 3y = 0, x_0 = 0.$

Substituting our power series in for  $y, y', y''$ :

$$2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} na_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Noting that in the second sum we could start at  $n = 0$ , since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} (2(k+2)(k+1)a_{k+2} + ka_k + 3a_k) x^k = 0$$

From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)} a_k$$

(b)  $(1-x)y'' + xy' - y = 0, x_0 = 0$

Substituting our power series in for  $y, y', y''$ :

$$(1-x) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

The two middle sums can have their respective index taken down by one (so that formally the series would start with  $0x^0$ ):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now make all the indices the same. To do this, in the first sum make  $k = n - 2$ , in the second sum take  $k = n - 1$ . Doing this and collecting terms:

$$\sum_{k=0}^{\infty} ((k+2)(k+1)a_{k+2} - (k+1)k a_{k+1} + (k-1)a_k) x^k = 0$$

So we get the recursion:

$$a_{k+2} = \frac{(k+1)k a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c)  $y'' - xy' - y = 0$ ,  $x_0 = 1$

Done in class;

$$a_{n+2} = \frac{1}{n+2} (a_{n+1} + a_n)$$

18. Exercises with the table:

(a) Use table entries 5 and 14 to prove the formula for 9.

SOLUTION: Prove formula #9 using 5 and 14:

$$\mathcal{L}(e^{at} \sin(bt)) = F(s-a)$$

where

$$F(s) = \mathcal{L}(\sin(bt)) = \frac{b}{s^2 + b^2} \Rightarrow \frac{b}{(s-a)^2 + b^2}$$

Therefore,

$$\mathcal{L}(e^{at} \sin(bt)) = \frac{b}{(s-a)^2 + b^2}$$

(b) Show that you can use table entry 19 to find the Laplace transform of  $t^2\delta(t-3)$  (verify your answer using a property of the  $\delta$  function).

SOLUTION: Using Entry 19, the Laplace transform of  $t^2\delta(t-3)$  is the second derivative of the Laplace transform of  $\delta(t-3)$ . That is, using

$$F(s) = e^{-3s}$$

then

$$\mathcal{L}(t^2\delta(t-3)) = F''(s) = 9e^{-3s}$$

And this is the same as:

$$\int_0^{\infty} e^{-st} t^2 \delta(t-3) dt = 9e^{-3s}$$

- (c) Prove (using the definition of  $\mathcal{L}$ ) table entries 12 and 13.

SOLUTION: 12 is a special case of 13, so we prove 13 using the definition:

$$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-st}u_c(t)f(t-c)dt = \int_c^\infty e^{-st}f(t-c)dt$$

We want this answer to be the following (with a different variable of integration):

$$e^{-cs}F(s) = e^{-cs} \int_0^\infty e^{-sw}f(w)dw = \int_0^\infty e^{-s(w+c)}f(w)dw$$

We can connect the two by taking  $w = t - c$  (so that  $t = w + c$ ), and then (remember to change the bounds!):

$$\int_c^\infty e^{-st}f(t-c)dt = \int_0^\infty e^{-s(w+c)}f(w)dw$$

And we're done.

- (d) Prove (using the definition of  $\mathcal{L}$ ) a formula (similar to 18) for  $\mathcal{L}(y'''(t))$ .

SOLUTION: I wanted you to recall how we got those definitions in the past (integrating by parts):

$$\mathcal{L}(y'''(t)) = \int_0^\infty e^{-st}y'''(t)dt$$

Integration by parts using a table:

$$\begin{array}{rcl} + & e^{-st} & y'''(t) \\ - & -se^{-st} & y''(t) \\ + & s^2e^{-st} & y'(t) \\ - & -s^3e^{-st} & y(t) \end{array} \Rightarrow \left( e^{-st}(y''(t) + sy'(t) + s^2y(t)) \right) \Big|_{t=0}^\infty + s^3 \int_0^\infty e^{-st}y(t)dt$$

At infinity, these terms all go to zero (otherwise, the Laplace transform wouldn't exist), so we get:

$$s^3 - (y''(0) + sy'(0) + s^2y(0)) = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

19. Find the first 5 terms of the power series solution to  $e^xy'' + xy = 0$  if  $y(0) = 1$  and  $y'(0) = -1$ .

Compute the derivatives directly, then (don't forget to divide by  $n!$ ):

$$y(x) = 1 - x - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \dots$$

20. Find the radius of convergence for the following series:

(a)  $\sum_{n=1}^\infty \sqrt{n}x^n$

SOLUTION:

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}}|x| = |x|$$

So by the ratio test, the series will converge (absolutely) if  $|x| < 1$  (so the radius is 1).

$$(b) \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}} (x+3)^n$$

SOLUTION: Simplifying the limit in the ratio test, we get

$$\lim_{n \rightarrow \infty} 2\sqrt{\frac{n}{n+1}} |x+3| = 2|x+3|$$

Therefore, by the ratio test, the series will converge absolutely if  $2|x+3| < 1$ , or if  $|x+3| < 1/2$  (and this is our radius). For the interval of convergence, we have to check the points  $x = -7/2$  and  $x = -5/2$  separately. For  $x = -7/2$ , the series diverges ( $p$ -test), and for  $x = -5/2$ , the series converges by the alternating series test.

*NOTE:* If you don't recall those tests, you probably ought to review them, but I won't make you recall them for the exam this week.

$$(c) \sum_{n=1}^{\infty} \frac{n! x^n}{n^n} \text{ (A little tricky)}$$

SOLUTION: This one is a little tricky because we need to recall the definition of  $e$ :

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

In this case, using the Ratio Test, we have:

$$\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} |x| = \lim_{n \rightarrow \infty} \frac{(n+1)n^n}{(n+1)(n+1)^n} |x| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n |x| = \frac{|x|}{e}$$

so the radius of convergence is  $e$ .

$$(d) \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$$

SOLUTION: This is definitely similar to problems on exams/quizzes. The Ratio Test simplifies to:

$$\frac{1}{5} \lim_{n \rightarrow \infty} \frac{n}{n+1} |3x-2| = \frac{|3x-2|}{5}$$

To converge absolutely,  $|3x-2| < 5$ . To get the radius of convergence, we need to have the form  $|x-a| < \rho$ , so in this case, we simplify to get:

$$3\left|x - \frac{2}{3}\right| < 5 \quad \Rightarrow \quad \left|x - \frac{2}{3}\right| < \frac{5}{3}$$

Now we have to check the endpoints separately, which are  $x = -1$  and  $x = 7/3$ :

- At  $x = -1$ , the sum becomes:

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating harmonic series, which converges (but not absolutely).

- At  $x = 7/3$ , the sum becomes a harmonic series, which diverges.

The interval of convergence is:  $[-1, \frac{7}{3})$