Exercise Set 2 (HW for 7.3, 7.5)

In the homework set, we will see *two* alternative ways of solving a simple first order system of differential equations: One way will be via substitution to a single second order equation (then use Chapter 3 methods). A second way will be to try to get an expression for dy/dx, then solve (using methods from Chapter 2). The standard approach is to use eigenvalues and eigenvectors, and we'll practice with those below, too. You should also log into our class website and look at the Java demos- I think they give a nice geometric explanation of "eigenvector" and "eigenvalue".

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

2. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if A is given below:

   (a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

3. For each matrix, find the eigenvalues and eigenvectors (these are selected from 16-23, p. 384 in the textbook). Note that they could be complex, and the matrix A may have complex numbers. Try the last one to see if you can do it!

   (a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

   (f) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

4. For each system below, find y as a function of x by first writing the differential equation as dy/dx.

   (a) $\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$

   (b) $\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned}$

   (c) $\begin{aligned} x' &= -(2x+3) \\ y' &= 2y - 2 \end{aligned}$

   (d) $\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned}$

5. For each given λ and $\mathbf{v}$, find an expression for the vector: $\mathrm{Im}(e^{\lambda t}\mathbf{v})$:

   (a) $\lambda = 3i$, $\mathbf{v} = [1-i, 2i]^T$

   (b) $\lambda = 1+i$, $\mathbf{v} = [i, 2]^T$

6. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the x_1, x_2 plane). Identify the origin as a *sink*, *source* or *saddle*:

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -6 & 10 \\ -2 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 8 & 6 \\ -15 & -11 \end{bmatrix}$