# Solutions for Exercise Set 2

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

SOLUTION: First, we compute $\mathbf{x}'$, then we'll compare it to $A\mathbf{x}$:

- For $\mathbf{x}'$, we have

$$-C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- For $A\mathbf{x}$, we have:

$$C_1 e^{-t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$

$$C_1 e^{-t} \begin{bmatrix} -1 \\ -2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = -C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Convert each of the systems $\mathbf{x}' = A\mathbf{x}$ into a single second order differential equation, and solve it using methods from Chapter 3, if $A$ is given below:

(a) $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

SOLUTION: We use the substitution from the first equation, $x_2 = \frac{1}{2}(x_1' - x_1)$ into the second equation:

$$\frac{1}{2}(x_1'' - x_1') = -5x_1 - \frac{1}{2}(x_1' - x_1) \quad \Rightarrow \quad x_1'' + 9x_1 = 0$$

We have two complex roots to the characteristic equation, $\lambda = \pm 3i$, so

$$x_1(t) = C_1 \cos(3t) + C_2 \sin(3t) \quad \Rightarrow \quad x_2(t) = \frac{1}{2}(x_1' - x_1)$$

which simplifies to

$$x_2(t) = C_1(-\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t)) + C_2(\frac{3}{2}\cos(3t) - \frac{1}{2}\sin(3t))$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

SOLUTION: Using the first equation to solve for $x_2 = x_1' - x_1$, substitute into the second to get:

$$x_1'' - x_1' = 4x_1 + x_1' - x_1 \quad \Rightarrow \quad x_1'' - 2x_1' - 3x_1 = 0$$

From solving the characteristic equation, $r = -1, 3$ so that

$$x_1 = C_1 e^{3t} + C_2 e^{-t}$$

Use the substitution to find $x_2 = x_1' - x_1$:

$$x_2 = 2C_1 e^{3t} - 2C_2 e^{-t}$$

(c) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

SOLUTION: In this case, the second equation is easier to use for the substitution: $x_1 = x_2' + x_2$, so that the first equation becomes:

$$x_1'' + x_2' = 3x_2' + 3x_2 - 4x_2 \quad \Rightarrow \quad x_2'' - 2x_2' + x_2 = 0$$

We have a double root: $r = 1, 1$. Thus,

$$x_2 = e^t (C_1 + C_2 t)$$

Substitute this into the equation for $x_1$:

$$x_1 = x_2' + x_2 = e^t(2C_1 + C_2(2t + 1))$$

3. For each matrix, find the eigenvalues and eigenvectors:

(a) $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

SOLUTION: The characteristic equation is

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad \lambda = 2, 4$$

For $\lambda = 2$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to

$$3v_1 - v_2 = 0 \quad \Rightarrow \quad \begin{aligned} v_1 &= v_1 \\ v_2 &= 3v_1 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

For $\lambda = 4$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to:

$$v_1 - v_2 = 0 \quad \Rightarrow \quad \begin{aligned} v_1 &= v_2 \\ v_2 &= v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

SOLUTION: The characteristic equation is

$$\lambda^2 - 2\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda = 1 \pm 2i$$

For $\lambda = 1 + 2i$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to

$$\begin{aligned} (2 - 2i)v_1 - 2v_2 &= 0 \\ 4v_1 - (2 + 2i)v_2 &= 0 \end{aligned}$$

It may not look like these are the same equation, but if you multiply the first equation by $2+2i$, you will get the second. Therefore, we can just use one of them- Using the first equation, we get:

$$v_2 = (1-i)v_1 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = v_1 \\ v_2 & = (1-i)v_1 \end{array} \quad \Rightarrow \quad \mathbf{v} = \left[ \begin{array}{r} 1 \\ 1-i \end{array} \right]$$

We don't need to solve for the second eigenvalue and eigenvector- They are simply the complex conjugates:

$$\lambda_2 = 1 - 2i \qquad \mathbf{v}_2 = \left[ \begin{array}{r} 1 \\ 1+i \end{array} \right]$$

(c) $A = \left[ \begin{array}{rr} -2 & 1 \\ 1 & -2 \end{array} \right]$

SOLUTION: The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = -1, -3$$

For $\lambda = -1$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to

$$-v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = v_1 \\ v_2 & = v_1 \end{array} \quad \Rightarrow \quad \mathbf{v} = \left[ \begin{array}{r} 1 \\ 1 \end{array} \right]$$

For $\lambda = -3$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to:

$$v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = -v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \left[ \begin{array}{r} -1 \\ 1 \end{array} \right]$$

(d) $A = \left[ \begin{array}{rr} 1 & i \\ -i & 1 \end{array} \right]$

SOLUTION: The characteristic equation is

$$\lambda^2 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = 0, 2$$

For $\lambda = 0$, the system may not look like the same equation, but they are (multiply the first equation by $-i$ to get the second):

$$v_1 + iv_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = -iv_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \left[ \begin{array}{r} -i \\ 1 \end{array} \right]$$

For $\lambda = 2$, we get:

$$-v_1 + iv_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = iv_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \left[ \begin{array}{r} i \\ 1 \end{array} \right]$$

(e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

SOLUTION: The characteristic equation is

$$\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda = 2, -2$$

For $\lambda = 2$, the system $(A - \lambda I)\mathbf{v} = 0$ reduces to

$$-v_1 + \sqrt{3}v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = \sqrt{3}v_1 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

For $\lambda = -2$, taking the second equation, we get

$$\sqrt{3}v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = v_1 \\ v_2 & = -\sqrt{3}v_1 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

(f) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

SOLUTION: This one is a little harder to do (only do it if you've had Math 300). However, you should find that the characteristic equation actually factors to:

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

For $\lambda = 1$, $\mathbf{v} = [-1, 0, 1]^T$. For $\lambda = 2$, we get $\mathbf{v} = [-2, 1, 0]^T$ and for $\lambda = 3$, $\mathbf{v} = [0, -1, 1]^T$.

4. For each system below, find $y$ as a function of $x$ by first writing the differential equation as $dy/dx$.

(a)

$$\begin{array}{ll} x' & = -2x \\ y' & = y \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{2x}$$

$$\frac{1}{y}dy = -\frac{1}{2} \cdot \frac{1}{x}\,dx \quad \Rightarrow \quad \ln|y| = -\frac{1}{2}\ln|x| + C \quad \Rightarrow \quad y = \frac{A}{\sqrt{x}}$$

(b)

$$\begin{array}{ll} x' & = y + x^3y \\ y' & = x^2 \end{array} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

$$y\,dy = \frac{x^2}{1+x^2}\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \ln|1+x^3| + C$$

(c)

$$\begin{aligned} x' &= -(2x+3) \\ y' &= 2y-2 \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2(y-1)}{2x+3}$$

$$\frac{1}{y-1}\,dy = \frac{-2}{2x+3}\,dx \quad \Rightarrow \quad \ln|y-1| = \ln\left(\frac{1}{2x+3}\right) + C \quad \Rightarrow$$

$$y = \frac{A}{2x+3} + 1$$

(d)

$$\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad y\,dy = -x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1 \quad \Rightarrow \quad x^2 + y^2 = C_2$$

5. For each given $\lambda$ and $\mathbf{v}$, find an expression for the vector: $\text{Im}(e^{\lambda t}\mathbf{v})$:

(a) $\lambda = 3i$, $\mathbf{v} = [1-i, 2i]^T$
SOLUTION: This one we'll do in detail, the next one is similar:

$$(\cos(3t) + i\sin(3t))\begin{bmatrix} 1-i \\ 2i \end{bmatrix} = \begin{bmatrix} (\cos(3t) - \sin(3t)) + i(\sin(3t) - \cos(3t)) \\ -2\sin(3t) + i(2\cos(3t)) \end{bmatrix}$$

Therefore, the imaginary part is:

$$\text{Im}(e^{\lambda t}\mathbf{v}) = \begin{bmatrix} \sin(3t) - \cos(3t) \\ 2\cos(3t) \end{bmatrix}$$

(b) $\lambda = 1 + i$, $\mathbf{v} = [i, 2]^T$
SOLUTION:

$$\text{Im}(e^{\lambda t}\mathbf{v}) = e^t \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}$$

6. Give the general solution to each system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, and sketch a phase plane (solutions in the $x_1, x_2$ plane). Identify the origin as a *sink*, *source* or *saddle*:

(a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$
SOLUTION:

$$C_1 e^{-4t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The origin is a saddle (the two eigenvalues are opposite in sign).

(b) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

SOLUTION:

$$C_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The origin is a source since both eigenvalues are positive.

(c) $A = \begin{bmatrix} -6 & 10 \\ -2 & 3 \end{bmatrix}$

SOLUTION:

$$C_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The origin is a sink since both eigenvalues are negative.

(d) $A = \begin{bmatrix} 8 & 6 \\ -15 & -11 \end{bmatrix}$

SOLUTION:

$$C_1 e^{-t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

And the origin is again a sink.