Integration By Parts- Via a Table

Typically, integration by parts is introduced as:

$$\int u \, dv = uv - \int v \, du$$

We want to be able to compute an integral using this method, but in a more efficient way. Consider the following table:

$$\int u \, dv \qquad \Rightarrow \qquad \begin{array}{c|c|c} + & u & dv \\ \hline - & du & v \end{array}$$

The first column switches $\pm$ signs, the second column differentiates u, and the third column antidifferentiates dv. We can write the result of integration as multiplying the sign, $+1$ times u then going down along a diagonal and multiplying by v. We then add the integral of the product going straight across.

Using this table, we can perform multiple integration by parts at one time. Consider this example, with the corresponding table:

$$\int t^2 \mathrm{e}^{-3t} \, dt \qquad \Rightarrow \qquad \begin{array}{c|c|c} + & t^2 & \mathrm{e}^{-3t} \\ \hline - & 2t & (-1/3)\mathrm{e}^{-3t} \\ \hline + & 2 & (1/9)\mathrm{e}^{-3t} \\ \hline - & 0 & (-1/27)\mathrm{e}^{-3t} \end{array}$$

Using the same pattern as before, but continuing through, we see that evidently:

$$\int t^2 \mathrm{e}^{-3t} \, dt = +t^2(-1/3)\mathrm{e}^{-3t} + (-2t)(1/9)\mathrm{e}^{-3t} + 2(-1/27)\mathrm{e}^{-3t} +$$

$$+ \int (-0 \cdot (-1/27)\mathrm{e}^{-3t} \, dt$$

Simplifying:

$$\int t^2 \mathrm{e}^{-3t} \, dt = -\mathrm{e}^{-3t} \left(\frac{1}{3}t^2 + \frac{2}{9}t + \frac{2}{27} \right)$$

Here are a couple more examples that usually require integration by parts:

$$\int \ln(x) \, dx \qquad \Rightarrow \qquad \begin{array}{c|c|c} + & \ln(x) & 1 \\ \hline - & 1/x & x \end{array}$$

so that:

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x$$

Another example, where we integrate by parts twice to get a similar integral on both sides of the equation:

$$\int \mathrm{e}^{-2t}\sin(3t)\ dt \qquad \Rightarrow \qquad \begin{array}{c|c|c} + & \sin(3t) & \mathrm{e}^{-2t} \\ \hline - & 3\cos(3t) & (-1/2)\mathrm{e}^{-2t} \\ \hline + & -9\sin(3t) & (1/4)\mathrm{e}^{-2t} \end{array}$$

So:

$$\int \mathrm{e}^{-2t}\sin(3t)\,dt = \sin(3t)(-1/2)\mathrm{e}^{-2t} - 3\cos(3t)(1/4)\mathrm{e}^{-2t} +$$

$$\int -9\sin(3t)(1/4)\mathrm{e}^{-2t}$$

Simplifying:

$$\int \mathrm{e}^{-2t}\sin(3t)\,dt = -\mathrm{e}^{-2t}\left(\frac{1}{2}\sin(3t) + \frac{3}{4}\cos(3t)\right) - \frac{9}{4}\int \mathrm{e}^{-2t}\sin(3t)\ dt$$

Now solve for the integral:

$$\frac{13}{4}\int \mathrm{e}^{-2t}\sin(3t)\,dt = -\mathrm{e}^{-2t}\left(\frac{1}{2}\sin(3t) + \frac{3}{4}\cos(3t)\right)$$

To finish, multiply both sides by 4/13.

Extra Practice:

Maple Commands are given so you can check your answer!

1. $\int \sqrt{x}\ln(x)\,dx$ `int(sqrt(x)*ln(x),x);`
2. $\int x^2\cos(3x)\,dx$ `int(x^2*cos(3*x),x);`
3. $\int t^3\mathrm{e}^{-2t}\,dt$ `int(t^3*exp(-2*t),t);`
4. $\int \mathrm{e}^{-2t}\sin(2t)\,dt$ `int(exp(-2*t)*sin(2*t),t);`
5. $\int \tan^{-1}(1/t)\,dt$ `int(arctan(1/t),t);`