Solutions: Section 2.1

1. Problem 1: See the Maple worksheet to get the direction field. You should see that it looks like all solutions are approaching some curve (maybe a line?) as $t \to \infty$. To be more analytic, let us solve the DE using the Method of Integrating Factors.

$$y' + 3y = t + \mathrm{e}^{-2t} \quad \Rightarrow \quad \mathrm{e}^{3t}\left(y' + 3y\right) = \mathrm{e}^{3t}\left(t + \mathrm{e}^{-2t}\right) \quad \Rightarrow \quad \left(\mathrm{e}^{3t}y(t)\right)' = t\mathrm{e}^{3t} + \mathrm{e}^{t}$$

Integrate both sides *Hint:* We need to use "integration by parts" to integrate $t\mathrm{e}^{3t}$. Using a table as in class:

$+$	t	e^{3t}
$-$	1	$(1/3)\mathrm{e}^{3t}$
$+$	0	$(1/9)\mathrm{e}^{3t}$

$$\Rightarrow \quad \int t\mathrm{e}^{3t}\,dt = \frac{1}{3}\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t}$$

Putting it all together,

$$\mathrm{e}^{3t}y(t) = \frac{1}{3}t\mathrm{e}^{3t} - \frac{1}{9}\mathrm{e}^{3t} + \mathrm{e}^{t} + C$$

so that

$$y(t) = \frac{1}{3}t - \frac{1}{9} + \mathrm{e}^{-2t} + C\mathrm{e}^{-3t}$$

Notice that the last two terms go to zero as $t \to \infty$, so we see that $y(t)$ does approach a line:

$$\frac{1}{3}t - \frac{1}{9}$$

as $t \to \infty$.

2. Problem 3: See Maple for the direction field. Very similar situation to Problem 1. Let's go ahead and solve:

$$y' + y = t\mathrm{e}^{-t} + 1$$

Multiply both sides by $\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{t}$:

$$\mathrm{e}^{t}\left(y' + y\right) = t + \mathrm{e}^{t} \quad \Rightarrow \quad \left(\mathrm{e}^{t}y(t)\right)' = t + \mathrm{e}^{t}$$

Integrate both sides:

$$\mathrm{e}^{t}y(t) = \frac{1}{2}t^2 + \mathrm{e}^{t} + C \quad \Rightarrow \quad y(t) = \frac{1}{2}t^2\mathrm{e}^{-t} + 1 + C\mathrm{e}^{-t}$$

This could be written as:

$$y(t) = 1 + \frac{t^2}{2\mathrm{e}^{t}} + \frac{C}{\mathrm{e}^{t}}$$

so that it is clear that, as $t \to \infty$, $y(t) \to 1$, which we also see in the direction field.

3. (Extra Practice, not assigned) Problem 11: See Maple for the direction field, where it looks like all solutions are approaching some periodic function as $t \to \infty$. Let's solve it:

$$y' + y = 5\sin(2t)$$

As in the last exercise, multiply both sides by e^t:

$$e^t(y' + y) = 5e^t \sin(2t) \quad \Rightarrow \quad \left(e^t y(t)\right)' = 5e^t \sin(2t)$$

To integrate the right-hand-side of this equation, we will need to use integration by parts twice. In tabular form:

$+$	e^t	$\sin(2t)$
$-$	e^t	$-(1/2)\cos(2t)$
$+$	e^t	$-(1/4)\sin(2t)$

$$\Rightarrow \quad \int e^t \sin(2t)\,dt = -\frac{1}{2}e^t\cos(2t) + \frac{1}{4}e^t \sin(2t) - \frac{1}{4}\int e^t \sin(2t)\,dt$$

Add the last integral to the left:

$$\frac{5}{4}\int e^t \sin(2t)\,dt = -\frac{1}{2}e^t\cos(2t) + \frac{1}{4}e^t\sin(2t)$$

so that:

$$\int e^t \sin(2t)\,dt = -\frac{2}{5}e^t\cos(2t) + \frac{1}{5}e^t\sin(2t) + C_1$$

Going back to the differential equation,

$$e^t y(t) = -2e^t\cos(2t) + e^t\sin(2t) + C_2$$

so that the general solution is:

$$y(t) = -2\cos(2t) + \sin(2t) + C_2 e^{-t}$$

We see that, as $t \to \infty$, $y(t)$ does indeed go to a periodic function.

In problems 13, 15, 16, solve the IVP. For these problems, I will leave the details out, but I will give the integrating factor. Be sure to ask in class if you're not sure how to solve them!

4. Problem 13: (You'll need to integrate by parts!)

$$y' - y = 2te^{2t} \qquad e^{\int p(t)\,dt} = e^{-t}$$

$$y(t) = e^{2t}(2t - 2) + 3e^t$$

5. Problem 15:

$$ty' + 2y = t^2 - t + 1$$

Be sure to put in standard form before solving:

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \qquad \mathrm{e}^{\int p(t)\,dt} = t^2$$

and

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

6. Problem 16: In this problem, the integrating factor is again t^2:

$$y' + \frac{2}{t} \cdot y = \frac{\cos(t)}{t^2} \qquad \Rightarrow \qquad y(t) = \frac{\sin(t)}{t^2}$$

7. Problem 30: Solve the IVP:

$$y' - y = 1 + 3\sin(t) \qquad y(0) = y_0$$

The integrating factor is e^{-t}, so that we get:

$$\left(y\mathrm{e}^{-t}\right)' = \mathrm{e}^{-t} + 3\mathrm{e}^{-t}\sin(t)$$

Integrate both sides. The term on the far right side of the equation requires integration by parts using a table (differentiate the middle column, antidifferentiate the last column):

$$\int \mathrm{e}^{-t}\sin(t)\,dt \quad \Rightarrow \quad \begin{array}{|c|c|c|} \hline + & \sin(t) & \mathrm{e}^{-t} \\ - & \cos(t) & -\mathrm{e}^{-t} \\ + & -\sin(t) & \mathrm{e}^{-t} \\ \hline \end{array}$$

so that

$$\int \mathrm{e}^{-t}\sin(t)\,dt = -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) - \int \mathrm{e}^{-t}\sin(t)\,dt$$

Add the last integral to both sides and divide:

$$2\int \mathrm{e}^{-t}\sin(t)\,dt = -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t)$$

Therefore,

$$\int \mathrm{e}^{-t}\sin(t)\,dt = -\frac{1}{2}\mathrm{e}^{-t}\left(\cos(t) + \sin(t)\right)$$

Going back to the DE, we have (remember to multiply by 3):

$$y\mathrm{e}^{-t} = -\mathrm{e}^{-t} - \frac{3}{2}\mathrm{e}^{-t}\left(\cos(t) + \sin(t)\right) + C$$

so that

$$y = -1 - \frac{3}{2}\left(\cos(t) + \sin(t)\right) + C\mathrm{e}^{t}$$

Therefore, the general solution is (using the initial condition):

$$y(t) = -1 - \frac{3}{2}(\cos(t) + \sin(t)) + \left(\frac{5}{2} + y_0\right) e^t$$

To keep the solution finite (or bounded) as $t \to \infty$, we must find y_0 so that the exponential term drops out- This means that $y_0 = -5/2$.

Problems 34-36: We're going backwards from a desired solution to the differential equation. We notice a couple of things about our previous work. If we want our solution to approach a given function $g(t)$, then we might write the full solution as

$$y(t) = g(t) + Ce^{-t}$$

Then, $y' = g'(t) - Ce^{-t}$, and $y' + y = g(t) - g'(t)$ is the differential equation.

8. For Problem 34, we want $y(t) \to 3$ as $t \to \infty$. Here are two possible ways of proceeding:

   - Suppose $y(t) = 3 + Ce^{-t}$ (so it looks a lot like the solutions we got for the previous HW problems). Then $y' = -Ce^{-t}$, and we see that:

     $$y' + y = 3$$

     (I'll leave the verification to you).
   - As another possible approach, we could take:

     $$y(t) = 3 + \frac{C}{t^2}$$

     Now, $y' = -2C/t^3$. We see that if we take ty' and add it to $2y$, the terms with C cancel and we're left with 6. Therefore, the ODE is:

     $$ty' + 2y = 6$$

     (I'll leave the verification to you).

9. Problem 35 is similar. There are many ways of constructing such a differential equation- It's easiest to start with a desired solution.

   If we would like $y(t) = 3 - t + Ce^{-3t}$, then $y' = -1 - 3Ce^{-3t}$, and:

   $$y' + 3y = 8 - 3t$$