Solutions: Section 2.2

- 2.2, 1: Give the general solution: $y' = x^2/y$

$$y\,dy = x^2\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

- 2.2, 3: Give the general solution to $y' + y^2\sin(x) = 0$.

  First write in standard form:

$$\frac{dy}{dx} = -y^2\sin(x) \quad \Rightarrow \quad -\frac{1}{y^2}\,dy = \sin(x)\,dx$$

  Before going any further, notice that we have divided by y, so we need to say that this is value as long as $y(x) \neq 0$. In fact, we see that the function $y(x) = 0$ IS a possible solution.

  With that restriction in mind, we proceed by integrating both sides to get:

$$\frac{1}{y} = -\cos(x) + C \quad \Rightarrow \quad y = \frac{1}{C - \cos(x)}$$

  Note: Don't forget to add the "C" at the right time- Right after integration.

- 2.2, 5 Hint: To integrate $\cos^2(x)$, use the identity $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

- 2.2, 7: Give the general solution:

$$\frac{dy}{dx} = \frac{x - \mathrm{e}^{-x}}{y + \mathrm{e}^{y}}$$

  First, note that dy/dx exists as long as $y \neq -\mathrm{e}^y$. With that requirement, we can proceed:

$$\left(y + \mathrm{e}^y\right)\,dy = \left(x + \mathrm{e}^{-x}\right)\,dx$$

  Integrating, we get:

$$\frac{1}{2}y^2 + \mathrm{e}^y = \frac{1}{2}x^2 - \mathrm{e}^{-x} + C$$

  In this case, we cannot algebraically isolate y, so we'll leave our answer in this form (we could multiply by two).

- 2.2, 9: Let $y' = (1-2x)y^2, \qquad y(0) = -1/6$.

  First, we find the solution. Before we divide by y, we should make the note that $y \neq 0$. We also see that $y(x) = 0$ is a possible solution (although NOT a solution that satisfies the initial condition).

  Now solve:

$$\int y^{-2}\,dy = \int (1-2x)\,dx \quad \Rightarrow \quad -y^{-1} = x - x^2 + C$$

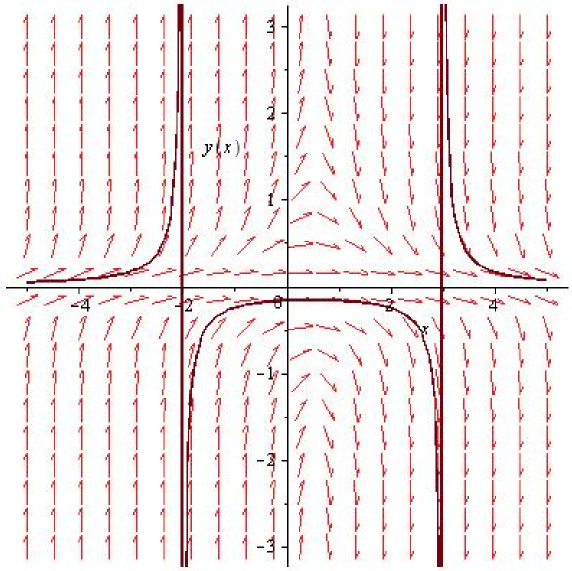

Figure 1: Graph for Exercise 9. Is the solution to the IVP represented by the black curve?

Solve for the initial value:

$$6 = 0 + C \Rightarrow C = 6$$

The solution is (solve for y):

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

The solution is valid only on $-2 < x < 3$, and we could plot this by hand, but for the plot, we can use Maple:

```
DE09:=diff(y(x),x)=(1-2*x)*(y(x))^2;
Y1:=dsolve( { DE09, y(0)=-1/6 },y(x));
with(plots):
plot(rhs(Y1),x=-5..5,y=-3..3);  #rhs means right hand side (of Y1)
```

NOTE: Here's an important question to think about. When we plot the graph of the solution, Maple includes the whole curve (minus the asymptotes at -2 and 3. Is this entire graph the solution?

- 2.2, 11: $x\,dx + y\mathrm{e}^{-x}dy = 0, \qquad y(0) = 1$

  To solve, first get into a standard form, multiplying by e^x, and integrate (integration by parts for the right hand side):

$$\int y\,dy = -\int x\mathrm{e}^x\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -x\mathrm{e}^x + \mathrm{e}^x + C$$

We could solve for the constant before isolating y:

$$\frac{1}{2} = 0 + 1 + C \quad C = -\frac{1}{2}$$

Now solve for y:

$$y^2 = 2e^x(x-1) - \frac{1}{2}$$

and take the positive root, since $y(0) = +1$.

$$y = \sqrt{2e^x(1-x) - 1}$$

The solution exists as long as:

$$2e^x(1-x) - 1 \geq 0$$

We use Maple to solve where this is equal to zero; from that, we see that $-1.678 \leq x \leq 0.768$

Here is the Maple code:

```
DE11:=x+y(x)*exp(-x)*diff(y(x),x)=0;
Y1:=dsolve({DE11,y(0)=1},y(x));
plot(rhs(Y1),x=-2..2);
evalf(solve(rhs(Y1)=0,x));
```

- 2.2, 16:

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3} \qquad y(0) = -\frac{1}{\sqrt{2}}$$

First, we notice that $y \neq 0$. Now separate the variables and integrate:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$$

This might be a good time to solve for C: $C = 1/4$, so:

$$y^4 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{4}$$

The right side of the equation seems to be a nice form. Try some algebra to simplify it:

$$\frac{1}{4}\left(x^4 + 2x^2 + 1\right) = \frac{1}{4}(x^2+1)^2$$

Now we can write the solution:

$$y^4 = \frac{1}{4}(x^2+1)^2 \quad \Rightarrow \quad y = -\frac{1}{\sqrt{2}}\sqrt{x^2+1}$$

This solution exists for all x, and the plot can be done in Maple:

```
DE14:=diff(y(x),x)=(x*(x^2+1))/(4*(y(x))^3);
Y1:=dsolve({DE14,y(0)=-1/sqrt(2)},y(x));
plot(rhs(Y1),x=-4..4);
```

- 2.2, 20: $y^2\sqrt{1-x^2}dy = \sin^{-1}(x)\,dx$ with $y(0)=1$.

  To put into standard form, we'll be dividing so that $x \neq \pm 1$. In that case,

$$\int y^2\,dy = \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$$

  The right side of the equation is all set up for a u, du substitution, with $u = \sin^{-1}(x)$, $du = 1/\sqrt{x^2-1}\,dx$:

$$\frac{1}{3}y^3 = \frac{1}{2}(\arcsin(x))^2 + C$$

  Solve for C, $\frac{1}{3} = 0 + C$ so that:

$$\frac{1}{3}y^3 = \frac{1}{2}\arcsin^2(x) + \frac{1}{3}$$

  Now,

$$y(x) = \sqrt[3]{\frac{3}{2}\arcsin^2(x) + 1}$$

  The domain of the inverse sine is: $-1 \leq x \leq 1$. However, we needed to exclude the endpoints. Therefore, the domain is:

$$-1 < x < 1$$

For Problems 31 and 35: We have a new class of differential equation called **homogeneous**. The idea is that the first order DE:

$$y' = f(x,y) = F(y/x) \doteq F(v)$$

Here, we substitute $v = y/x$ and see what we get- The hard part is to make the substitution for y'- Notice that $vx = y$, so $y' = v'x + v$. Substituting, we have:

$$y' = F(y/x) \quad \Rightarrow \quad v'x + v = F(v)$$

which is always a separable equation.

- Problem 31:

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

  Make the substitutions: $v = y/x$ and $y' = v'x + v$:

$$v'x + v = 1 + v + v^2 \quad \Rightarrow \quad x\frac{dv}{dx} = 1 + v^2 \quad \Rightarrow \quad \frac{dv}{1+v^2} = \frac{dx}{x}$$

Integrate both sides to get $\tan^{-1}(v) = \ln|x| + C$, and now we'll see if we can solve for y:

$$\tan^{-1}(y/x) = \ln|x| + C \quad \Rightarrow \quad y = x\tan(\ln|x| + C)$$

We have to be a bit careful about the domain for this function- Recall that $y = \tan(x)$ is invertible only if we restrict $-\pi/2 < x < \pi/2$ (and $y \in \mathbb{R}$). In this case, that means

$$-\frac{\pi}{2} < \ln|x| + C < \frac{\pi}{2} \quad \Rightarrow \quad -C - \frac{\pi}{2} < \ln|x| < -C + \frac{\pi}{2}$$

Exponentiating,

$$\mathrm{e}^{-c}\mathrm{e}^{-\pi/2} < |x| < \mathrm{e}^{-c}\mathrm{e}^{\pi/2}$$

We might go ahead and drop the absolute value at this point.

- Problem 35: Similar to 31,

$$\frac{dy}{dx} = \frac{x+3y}{x-y} = \frac{1+3(y/x)}{1-(y/x)}$$

Subsitute again, $v = y/x$, or $y = xv$, so $y' = v'x + v$:

$$v'x + v = \frac{1+3v}{1-v} \quad \Rightarrow \quad v'x = \frac{-v(1-v)}{1-v} + \frac{1+3v}{1-v} = \frac{v^2+2v+1}{1-v} = \frac{(1+v)^2}{1-v}$$

Now, let $u = 1 + v$ (so $v = u - 1$), and $du = dv$:

$$\int \frac{1-v}{(1+v)^2}\,dv = \int \frac{dx}{x} \quad \Rightarrow \quad \int \frac{2-u}{u^2} = \ln|x| + C \quad \Rightarrow \quad -2(1+v)^{-1} - \ln|1+v| = \ln|x| + C$$

Backsubstitute for v (and simplify):

$$\frac{-2x}{x+y} - (\ln|x+y| - \ln|x|) = \ln|x| + C \quad \Rightarrow \quad \frac{2x}{x+y} + \ln|x+y| = C_2$$

This solution is valid as long as $y \neq -x$. Is the function $y = -x$ a solution as well? Substitute into the DE, with $y' = -1$, we see that:

$$\frac{x+3y}{x-y} = \frac{-2x}{2x} = -1$$

so indeed it is.