Solutions: Section 2.4

1. Problems 1-6 ask you to apply the Existence and Uniqueness Theorem to a given linear ODE. Be sure to put the DE in standard form first! Some notes as you do this:

   - The interval is a single (connected) interval.
   - For theoretical reasons, our interval should be open (so it is possible to differentiate the function at each point in the domain).
   - The actual interval may be larger than the one guaranteed by the theorem (but we are looking for the one guaranteed by the theorem).

   For example, in Exercise 5:

$$(4-t^2)y' + 2ty = 3t^2 \qquad y(1) = -3 \quad \Rightarrow \quad y' + \frac{2t}{(2-t)(2+t)}y = \frac{3t^2}{4-t^2}$$

   The functions p, q are continuous on $(-\infty, -2) \bigcup (-2, 2) \bigcup (2, \infty)$. We want the interval containing $t = 1$, which is the middle interval.

2. Exercises 7-12 ask you to apply the general existence and uniqueness theorem. Sometimes it can be difficult to tell where a functions in the plane are continuous (especially if we had to use the definition), but we are looking for common constraints, like where the denominator is zero, or making sure the log or square root are defined.

   - Problem 7:

$$f(t,y) = \frac{t-y}{2t+5y} \qquad f_y = \frac{-7t}{(2t+5y)^2)}$$

     Therefore, f and f_y are both continuous for all (t, y) except those points along the line $2t + 5y = 0$, or $y = -2t/5$

   - Problem 9:

$$f(t,y) = \frac{\ln|ty|}{1-t^2+y^2} \qquad f_y = \frac{(1/y)(1-t^2+y^2) - \ln|ty|(2y)}{(1-t^2+y^2)^2}$$

     Therefore, the derivative did not add any new restrictions- f is continuous on the (t, y) plane except on the axes $t = 0$ and $y = 0$, and the curve $1 - t^2 + y^2 = 0$.

3. In problems 13-16, we solve the differential equation to determine the full interval on which solutions exist (and how they depend on the initial condition). We show in detail the solution to Exercise 13 and 14:

   - 2.4, 13:
     Given $y' = -\frac{4t}{y}$, we see that $f(t, y)$ from the Existence and Uniqueness theorem is

$$f(t,y) = -\frac{4t}{y} \qquad f_y(t,y) = \frac{4t}{y^2}$$

so the Existence and Uniqueness theorem holds as long as $y(0) = y_0 \neq 0$. Now, solving the IVP (it is separable):

$$y\,dy = -4t\,dt \quad \Rightarrow \quad \frac{1}{2}y^2 = -2t^2 + C_1 \quad \Rightarrow \quad y^2 = -4t^2 + C_2$$

Therefore, solving for the C_2, we get $C_2 = y_0^2$, and

$$y(t) = \pm\sqrt{y_0^2 - 4t^2}$$

This is valid as long as $y > 0$ and the expression under the radical sign is positive:

$$y_0^2 - 4t^2 > 0 \quad \Rightarrow \quad 4t^2 < y_0^2 \quad \Rightarrow \quad t^2 < \frac{y_0^2}{4}$$

This is true as long as (assuming $y_0 > 0$):

$$-\frac{y_0}{2} < t < \frac{y_0}{2}$$

If $y_0 < 0$, then put absolute value signs around y_0.

QUESTION: Don't we still get two solutions from $y = \pm\sqrt{y_0^2 - 4t^2}$?

ANSWER: Nope. If $y_0 < 0$, choose the negative root. If $y_0 > 0$, choose the positive root (Graphically, this is the upper or lower half of the ellipse $4t^2 + y^2 = y_0^2$).

- 2.4, 14 Given $y' = 2ty^2$, $y(0) = y_0$, we see it is separable:

$$\int y^{-2}\,dy = \int 2t\,dt \quad \Rightarrow \quad -\frac{1}{y} = t^2 + C$$

With the initial condition, $C = -1/y_0$, so:

$$y(t) = \frac{1}{(1/y_0) - t^2} = \frac{y_0}{1 - y_0 t^2}$$

This is valid as long as $y_0 \neq 0$. What if it is? Then we see that $y(t) = 0$ is the unique solution.

If $y_0 \neq 0$, then we continue by looking at where

$$1 - y_0 t^2 = 0 \qquad \Rightarrow \qquad t = \pm\frac{1}{\sqrt{y_0}}$$

This is valid only if $y_0 > 0$. If $y_0 < 0$, then the denominator, $1 - y_0 t^2$ is never zero (for any t). Thus, if $y_0 < 0$, the solution that we previously obtained is valid for all t.

The last case is where $y_0 > 0$. Since the initial time is $t_0 = 0$, then the solution $y(t)$ is only valid for:

$$-\frac{1}{\sqrt{y_0}} < t < \frac{1}{\sqrt{y_0}}$$

Summary: In this homework problem, we saw that the time interval on which the solution is valid depended greatly on the initial value of y,

- If $y_0 < 0$, $y(t)$ is valid for all time.
- If $y_0 = 0$, $y(t) = 0$ is the solution, valid for all time.
- If $y_0 > 0$, $y(t)$ is valid for a short segment of time, between $\pm 1/\sqrt{y_0}$.

4. Exercises 21-22 have solutions in the back of the text, but here is 22:

5. 2.4, 22: Part (a) involves substituting the expressions in to be sure the DE is satisfied. For part (b), we note that $t^2 + 4y \geq 0$ for the square root.

   In particular, if $y = 1 - t$, then this implies that

   $$1 - t \geq -\frac{1}{4}t^2$$

   which is true for all t (graph them), and the second function is clearly valid.

   Using the existence and uniqueness theorem, we see that $F_y(t, y) = \frac{1}{\sqrt{t^2+4y}}$, which does not exist at $(2, -1)$.

   The point of part (c) is to show that the second solution, $y_2(t)$ cannot be found as a constant multiple of the first (why? This will become clearer in Chapter 3).

   Clearly, no value of a constant in $ct + c^2$ can give you the function $-\frac{1}{4}t^2$.

6. Exercises 23-25: If you have had linear algebra, you might see that there is an underlying point to these- Linearity. If you have not, don't worry about it yet. However, some students are not sure about what constitutes an answer, so they are provided below:

   - Problem 23:

     (a) Show that e^{2t} and ce^{2t} (c any constant) are both solutions to the ODE: $y' - 2y = 0$.
     You can show this directly (by substitution), or by actually solving the DE. You should see that the general solution is $y(t) = Ae^{2t}$

     (b) Show that $\frac{1}{t}$ is a solution to $y' + y^2 = 0$, but $\frac{C}{t}$ is not.
     You can again show this directly (by substitution), or by actually solving the DE. If you solve it, you should get:

     $$y(t) = \frac{1}{t - C}$$

     (or $y(t) = 0$).

- Problem 24: To show this, first note that, if $y(t) = \phi(t)$ is a solution to $y' + p(t)y = 0$, then:
$$\phi' + p(t)\phi = 0$$
Now substitute $y(t) = c\phi$: $y' = c\phi'$, and:
$$c\phi' + p(t)c\phi = c\left(\phi' + p(t)\phi\right) = c \cdot 0 = 0$$
- Problem 25: Same idea as 24. Substitute the expression in to see what you get.
Assume that y_1 solves $y' + p(t)y = 0$. This means that $y_1' + p(t)y_1' = 0$.
Assume that y_2 solves $y' + p(t)y = g(t)$. This means that $y_2' + p(t)y_2 = g(t)$.
Now, substitute $y = y_1 + y_2$, $y' = y_1' + y_2'$ into the DE:
$$(y_1' + y_2') + p(t)(y_1 + y_2) = (y_1' + p(t)y_1) + (y_2' + p(t)y_2) = 0 + g(t) = g(t)$$

7. Exercises 27-31 focus on a class of DE's known as **Bernoulli** equations. Exercise 27 steps you through the process:
$$y' + p(t)y = q(t)y^n \quad \Rightarrow \quad \frac{y'}{y^n} + p(t)\frac{1}{y^{n-1}} = q(t)$$
This is "almost" a linear DE- Let $v = \frac{1}{y^{n-1}} = y^{1-n}$. Then
$$v' = (1-n)y^{1-n-1}y' = (1-n)\frac{y'}{y^n}$$
Therefore, if we multiply both sides by $1-n$, then we can substitute:
$$(1-n)\frac{y'}{y^n} + (1-n)p(t)\frac{1}{y^{n-1}} = (1-n)q(t) \quad \Rightarrow \quad v' + (1-n)v = (1-n)q(t)$$
which is a linear DE in v. Now we'll use this technique to solve **Exercise 28**, and 29 is similar:

Given $t^2y' + 2ty = y^3$, divide by t^2y^3 to get the equation in a form we can use:
$$\frac{y'}{y^3} + \frac{2}{t}\frac{1}{y^2} = \frac{1}{t^2}$$
And we'll substitute: $v = y^{-2}$, so that $v' = -2y^{-3}y'$. To get the substitution, multiply the DE by -2:
$$-2\frac{y'}{y^3} - \frac{4}{t}\frac{1}{y^2} = -\frac{2}{t^2} \quad \Rightarrow \quad v' - \frac{4}{t}v = -\frac{2}{t^2}$$
Now the integrating factor is t^{-4}, so that
$$v(t) = \frac{2}{5t} + Ct^4 \quad \Rightarrow \quad \frac{1}{y^2} = \frac{2 + C_2t^5}{5t} \quad \Rightarrow y(t) = \pm\sqrt{\frac{5t}{2 + C_2t^5}}$$

8. Exercise 33 gives us a chance to work with discontinuities (we will pick these up again in Chapter 6, Laplace Transforms). We can solve it now, though. In this case, $p(t)$ depends on time, so we can solve it in pieces:

$$y' + 2y = 0 \quad y(0) = 1 \quad 0 \le t \le 1 \qquad \text{And } y' + y = 0, t > 1$$

If we solve each of these, we get:

$$y(t) = \begin{cases} \mathrm{e}^{-2t} & \text{if } 0 \le t \le 1 \\ P\mathrm{e}^{-t} & \text{if } t > 1 \end{cases}$$

where P is any constant. If it is possible to find P so that y is continuous at all time, then we should go ahead and note that: For y to be continuous, we must have:

$$\mathrm{e}^{-2(1)} = P\mathrm{e}^{-1} \quad \Rightarrow \quad P = \mathrm{e}^{-1}$$

Therefore, for $t > 1$, we can write y as $\mathrm{e}^{-1}\mathrm{e}^{-t} = \mathrm{e}^{-(t+1)}$