Selected Solutions, Section 3.1

For Exercises 1 and 3, we practice using the ansatz and characteristic equation to solve the ODE. Here is #3 and #10:

3. $6y'' - y' - y = 0$

   SOLUTION: Using the ansatz $y = e^{rt}$, we obtain the characteristic equation, which factors (you could use the quadratic formula, but that does take extra time):

   $$6r^2 - r - 1 = 0 \quad \Rightarrow \quad (2r-1)(3r+1) = 0$$

   so we have two distince real solutions, $r = 1/2, -1/3$. In that case, the general solution is given by:

   $$y = C_1 e^{t/2} + C_2 e^{-t/3}$$

10. $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$

    SOLUTION: This problem adds the initial conditions, so we'll need to solve for C_1, C_2 after we get the general solution. Here we go!

    The characteristic equation is given below, which factors:

    $$r^2 + 4r + 3 = 0 \quad \Rightarrow \quad (r+1)(r+3) = 0 \quad \Rightarrow \quad r = -1, -3$$

    The general solution is

    $$y(t) = C_1 e^{-t} + C_2 e^{-3t}$$

    Substitute the initial conditions to solve for C_1, C_2:

    $$\begin{array}{rl} y(0) & = 2 \\ y'(0) & = -1 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} C_1 & +C_2 & = 2 \\ -C_1 & -3C_2 & = -1 \end{array}$$

    Sometimes it is convenient to use Cramer's Rule to solve this. The determinant of the coefficient matrix is $-3 + 1 = -2$, so we have

    $$C_1 = \frac{\begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix}}{-2} = \frac{5}{2} \qquad C_1 = \frac{\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}}{-2} = -\frac{1}{2}$$

    Additionally, the solution:

    $$y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

    will tend to zero as $t \to \infty$.

15-16. Very similar to 10.

17-18. The idea here is to go backwards from the characteristic equation to the differential equation:

$$ay'' + by' + cy = 0 \quad \Leftrightarrow \quad ar^2 + br + c = 0$$

For 17, the characteristic equation would be $(r-2)(r+3) = 0$, or $r^2 + r - 6 = 0$ (then get the DE). For 18, $(r + 1/2)(r + 2) = 0$ or multiply both sides by 2 to get rid of the fraction: $(2r + 1)(r + 2) = 0$ (expand it out to get the coefficients for the DE).

20. First solve the IVP:

$$2r^2 - 3r + 1 = 0 \quad \Rightarrow \quad (2r-1)(r-1) = 0 \quad \Rightarrow \quad r = 1, \frac{1}{2}$$

The general solution is: $y(t) = C_1 e^{t/2} + C_2 e^t$. Substituting the initial conditions, solve using elimination or Cramer's Rule:

$$\begin{array}{rl} C_1 + C_2 & = 2 \\ \frac{1}{2}C_1 + C_2 & = \frac{1}{2} \end{array} \quad \Rightarrow \quad C_1 = 3 \qquad C_2 = -1$$

Therefore, $y(t) = 3e^{t/2} - e^t$.

Now, we can find the maximum by looking at the derivative and setting it to zero:

$$y' = \frac{3}{2}e^{t/2} - e^t = 0 \quad \Rightarrow \quad \frac{3}{2} = e^{t/2} \quad \Rightarrow \quad \ln\left(\frac{3}{2}\right) = \frac{t}{2}$$

Therefore, $t = 2\ln(3/2) \approx 0.8109$. We can check the sign of the derivative to see if this is a maximum (the first derivative test):

y'	$+$	$-$
	$t < 2\ln(3/2)$	$t > 2\ln(3/2)$

Therefore, we have a global (or absolute) maximum at $t = 2\ln(3/2)$.

Furthermore, we can find where the solution is zero by setting $y(t) = 0$ and solving for t- We get $t = 2\ln(3)$.

21. Go to the characteristic equation first to get the general solution:

$$r^2 - r - 2 = 0 \quad \Rightarrow \quad (r-2)(r+1) = 0 \quad r = 2, -1$$

Therefore, the general solution is:

$$y(t) = C_1 e^{2t} + C_2 e^{-t}$$

Solve using the ICs: $y(0) = \alpha$ and $y'(0) = 2$, we have the following system (solve it using Cramer's Rule):

$$\begin{array}{rl} C_1 + C_2 & = \alpha \\ 2C_1 - C_2 & = 2 \end{array} \quad \Rightarrow \quad C_1 = \frac{\alpha + 2}{3} \qquad C_2 = \frac{2\alpha - 2}{3}$$

Now, if C_1 is anything but zero, over time, the term e^{2t} will take $y(t)$ to either $\pm\infty$ (depending on the sign of C_1), and so for the solution to approach either zero or stay bounded, the constant C_1 must be zero- In terms of α, we must have $\alpha = -2$.

22. This one is very similar to 21.

23.

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1) = 0$$

Before going farther, we note that given $ay'' + by' + cy = 0$, the solution to the characteristic equation is given by the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this particular case,

$$b^2 - 4ac = (2\alpha - 1)^2 - 4\alpha(\alpha - 1) = 1$$

The solutions to the characteristic equation are thus

$$r = \frac{(2\alpha - 1) \pm 1}{2} \quad \Rightarrow \quad r = \alpha, \alpha - 1$$

We have three cases to consider when looking at whether the solutions are bounded:

- For all solutions to go to zero as $t \to \infty$ (no matter the choice of initial conditions), both values of r must be negative:

  $$\alpha < 0 \quad \text{and} \quad \alpha < 1$$

  or $\alpha < 0$.

- If one value of r is positive and one value of r is negative, then some of the solutions will go to zero, some become unbounded (that is, we can find initial conditions that will keep the solution bounded, and some that will not).

- To guarantee that all solutions become unbounded (regardless of initial condition), both values of r have to be positive:

  $$\alpha > 0 \quad \text{and } \alpha > 1$$

  or, $\alpha > 1$.