Trigonometry Review: Cosine Sums

Recall the cosine sum formula:

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

Using this, we see that:

$$(R\cos(\delta))\cos(\omega t) + (R\sin(\delta))\sin(\omega t) = R\cos(\omega t - \delta)$$

This implies that we can write the following sum as a single cosine function:

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta) \quad \Leftrightarrow \quad A = R\cos(\delta) \qquad B = R\sin(\delta)$$

or, given A, B then:

$$R = \sqrt{A^2 + B^2} \quad \text{and} \quad \tan(\delta) = \frac{B}{A}$$

These computations will be useful for us in analyzing the solution to the DE when $r = \alpha \pm \beta i$, since the general form of the solution is:

$$e^{\alpha t}\left(C_1\cos(\beta t) + C_2\sin(\beta t)\right) = Re^{\alpha t}\cos(\beta t - \delta)$$

Examples

1. Write the solution to $y'' + 2y' + 6y = 0$, $y(0) = 1$, $y'(0) = 1$ in the form $Re^{\alpha t}\cos(\beta t - \delta)$:

   SOLUTION: Solve the characteristic equation:

   $$r^2 + 2r + 6 = 0 \quad \Rightarrow \quad (r^2 + 2r + 1) + 5 = 0 \quad \Rightarrow \quad r = -1 \pm \sqrt{5}\,i$$

   The general form of the solution is:

   $$y(t) = e^{-t}\left(C_1\cos(\sqrt{5}\,t) + C_2\sin(\sqrt{5}\,t)\right)$$

   Solving for the initial conditions (a bit of algebra involved here!):

   $$C_1 = 1 \qquad C_2 = \frac{2}{\sqrt{5}}$$

   Therefore,

   $$R = \sqrt{1 + \frac{4}{5}} = \frac{3}{\sqrt{5}}$$

   And

   $$\delta = \tan^{-1}(2/\sqrt{5}) \approx 0.7297$$

   Therefore,

   $$y(t) = \frac{3}{\sqrt{5}}e^{-t}\cos\left(\sqrt{5}\,t - 0.7297\right)$$

   Now it is easy to analyze the solution as a periodic function being modified (dampened by the exponential). The periodic function has a period of $2\pi/\sqrt{5}$ and an amplitude of $3/\sqrt{5}$.

2. Example: Same instructions as the previous problem, with $y'' + 192y = 0$, $y(0) = 1/6$ and $y'(0) = -1$:

$$r^2 + 192 = 0 \quad \Rightarrow \quad r = \pm 8\sqrt{3}\, i$$

The general form of the solution is (with $\alpha = 0$ in the exponential):

$$y(t) = C_1 \cos(8\sqrt{3}\, t) + C_2 \sin(8\sqrt{3}\, t)$$

Using the initial conditions, we see that:

$$C_1 = 1/6 \quad C_2 = -\frac{1}{8\sqrt{3}}$$

so that

$$y(t) = \frac{1}{6}\cos(8\sqrt{3}\, t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}\, t)$$

Rewriting this as a single cosine function,

$$R = \sqrt{\frac{1}{36} + \frac{1}{192}} \approx 0.186$$

and

$$\tan(\delta) = -\frac{\sqrt{3}}{4}$$

There are two solutions to this- in Quadrant II or Quadrant IV. Since $C_1 = R\cos(\delta) > 0$ and $C_2 = R\sin(\delta) < 0$, take the one in the fourth quadrant (returned by the calculator):

$$\delta \approx -0.40864$$

Exercises

1. Write the expression $3\cos(5t) - 2\sin(5t)$ in the form $R\cos(\omega t - \delta)$.
2. Write the expression $-\cos(2t) + \sin(2t)$ in the form $R\cos(\omega t - \delta)$.
3. Write the expression $-\cos(t) - 3\sin(t)$ in the form $R\cos(\omega t - \delta)$.
4. Write the expression $\mathrm{e}^{-t}\cos(2t) + 2\mathrm{e}^{-t}\sin(2t)$ in the form $R\mathrm{e}^{\alpha t}\cos(\omega t - \delta)$, then sketch its graph.
5. Given $y'' + 64y = 0$, $y(0) = -1$, $y'(0) = -1$, find the solution, its amplitude, period and phase shift.
6. Given the general equation: $my'' + \gamma y = 0$ with m, γ positive real numbers, find the natural frequency (a.k.a. circular frequency) of the solution.