

## Selected Solutions: 3.4 (Repeated Roots)

In Exercises 1-14, we're simply applying the "summary", p. 170. That is, the exercises have mixed up the cases where we have distinct real roots (3), complex roots (5), and repeated roots (1, 6, 8, 12, 14).

16. Solve the IVP:

$$y'' - y' + \frac{1}{4}y = 0 \quad y(0) = 2, \quad y'(0) = b$$

From the characteristic equation,  $r = 1/2$  is a repeated root. Therefore, the general solution is:

$$y = e^{t/2}(C_1 + C_2t)$$

Solving for the coefficients,

$$y = e^{t/2}(2 + (b-1)t)$$

For large  $t$ , the term  $(b-1)te^{t/2}$  will dominate. Therefore, as  $t \rightarrow \infty$ , if  $b > 1$ , this term goes to  $+\infty$ , and if  $b < 1$ , then this term goes to  $-\infty$ .

18. Solving the differential equation with the given initial values, we get:

$$y = e^{-\frac{2}{3}t} \left( \alpha + \frac{2\alpha - 3}{3}t \right)$$

By an argument similar to the previous problem, the critical value of  $\alpha$  is the one that makes  $2\alpha - 3 = 0$ . Therefore,  $\alpha = \frac{3}{2}$ .

20. Done in class in the more general form of  $ay'' + by' + cy = 0$ . In this special case,

(a) Char Eqn:  $r^2 + 2ar + a^2 = 0 \Rightarrow (r + a)^2 = 0$ , so  $r = -a$  is a double root.

(b) The Wronskian:

$$W(y_1, y_2) = Ce^{-\int p(t) dt} = Ce^{-2at}$$

(c) Let  $y_1 = e^{-at}$ . Find  $y_2$  using the Wronskian:

$$\begin{vmatrix} e^{-at} & y_2 \\ -ae^{-at} & y_2' \end{vmatrix} = e^{-at} (y_2' + ay_2)$$

Setting the two expressions for the Wronskian equal to each other enables us to find  $y_2$ :

$$e^{-at} (y_2' + ay_2) = Ce^{-2at} \Rightarrow y_2' + ay_2 = Ce^{-at}$$

The integrating factor is  $e^{at}$ . Multiply both sides by it, and go through our usual method for solving:

$$(y_2 e^{at})' = C \Rightarrow y_2 e^{at} = Ct + C_2 \Rightarrow y_2 = Cte^{-at} + C_2 e^{-at}$$

The last term is already present as  $y_1$ , so we take  $y_2$  to be  $te^{-at}$ .

21. We are told that  $e^{r_1 t}$  and  $e^{r_2 t}$  are solutions to the DE:

$$ay'' + by' + cy = 0$$

where  $r_1 \neq r_2$ .

Is the following function also a solution?

$$\phi = \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = \frac{1}{r_2 - r_1} e^{r_2 t} - \frac{1}{r_2 - r_1} e^{r_1 t}$$

Yes, since it is of the form  $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ .

Now use L'Hospital's rule to evaluate this solution as  $r_2 \rightarrow r_1$ . Note that we'll be differentiating with respect to the variable that is changing,  $r_2$ . Therefore, treat  $r_1$  and  $t$  as constants:

$$\lim_{r_2 \rightarrow r_1} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} = \lim_{r_2 \rightarrow r_1} \frac{te^{r_2 t} - 0}{1 - 0} = te^{r_1 t}$$

*Exercises 24 and 25 practice the method of Reduction of Order to get a second solution from the first (as discussed in the text). Try one of them; there are other ways of doing this, but it's good practice to be able to use several techniques.*

24. Using the ansatz  $y_2 = v(t)y_1(t) = vt$ , we substitute

$$y_2' = v't + v \quad y_2'' = v''t + 2v'$$

so that

$$t^3 v'' + 4t^2 v' = 0$$

This is linear in  $v'$ , so solve this to get

$$v' = C_1 t^{-4} \quad \Rightarrow \quad v = C_2 t^{-3}$$

Since the constant is arbitrary, we choose a nice one-  $C_2 = 1$ , and

$$y_2 = v(t)y_1(t) = \frac{t}{t^3} = t^2$$

(You could double check your work by changing the ansatz to  $y = t^r$  and see if you get the same thing).

25. Same idea as before. You should end up with:

$$tv'' + v' = 0$$

which we solve to get

$$v' = \frac{C}{t} \quad \Rightarrow \quad v = C \ln(t)$$

so that (choose  $C = 1$ ),

$$y_2 = v(t)y_1(t) = \frac{\ln(t)}{t}$$

34. This is the technique we used in class to find a second solution- Compute the Wronskian in two ways and equate them.
38. *Side Remark: This is a good problem to work through, since many models in physics reduce to this form.*

If  $a, b, c$  are positive constants, then we show that all solutions to  $ay'' + by' + cy = 0$  tend to zero as  $t \rightarrow \infty$ .

We need to consider our three cases:

- If  $b^2 - 4ac > 0$  then we need to consider the sign of  $-b \pm \sqrt{b^2 - 4ac}$ . If  $a, b, c$  are positive then

$$b > \sqrt{b^2 - 4ac}$$

since we are subtracting from  $\sqrt{b^2}$ . In that case,  $r_1$  and  $r_2$  are both negative, and

$$C_1 e^{r_1 t} + C_2 e^{r_2 t} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

- If  $b^2 - 4ac < 0$ , then:

$$r = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}$$

And the general solution is:

$$y = e^{-(b/2a)t} \left( C_1 \cos \left( \frac{\sqrt{4ac - b^2}}{2a} \right) + C_2 \sin \left( \frac{\sqrt{4ac - b^2}}{2a} \right) \right)$$

The sum of a sine and cosine are bounded, so the exponential term in front drives the solution to zero as  $t \rightarrow \infty$ .

- If  $b^2 - 4ac = 0$ , then  $r = -b/2a$  and the general solution is:

$$y(t) = e^{-(b/2a)t} (C_1 + C_2 t)$$

From L'Hospital's rule,

$$\lim_{t \rightarrow \infty} \frac{C_1 + C_2 t}{e^{(b/2a)t}} = \lim_{t \rightarrow \infty} \frac{C_2}{(b/2a)e^{(b/2a)t}} = 0$$

*Extra:* From our answers to 24, 25 and the last section, can we try to summarize the solutions to:

$$t^2 y'' + \alpha t y' + \beta y = 0$$

If we use the ansatz  $y = t^r$ , we have three cases (just as we have for  $ay'' + by' + cy = 0$ ).