Selected Solutions: 3.5 (Undetermined Coefficients)

In Exercises 1-10, we want to apply the ideas from the table to specific DEs, and solve for the coefficients as well. If you prefer, you might start with Exercises 19-26 first, since there is less algebra (then come back to these afterwards).

1. We have distinct roots, and no part of the homogeneous solution is included in the ansatz for the particular solution. That is,

$$r^2 - 2r - 3 = 0 \quad \Rightarrow \quad r = 3, -1$$

Therefore, $y_h = C_1 e^{3t} + C_2 e^{-t}$. In this exercise, $g(t) = 3e^{2t}$, so our ansatz for the particular solution is

$$y_p = Ae^{2t}$$

Substituting this into the DE, we get

$$4Ae^{2t} - 2(2Ae^{2t}) - 3Ae^{2t} = 3e^{2t}$$

From which we see that $A = -1$. Therefore, the overall general solution is given by:

$$y(t) = y_h(t) + y_p(t) = C_1 e^{3t} + C_2 e^{-t} - e^{2t}$$

4. In this case, we will get distinct roots, and there will be some overlap between the homogeneous solution and the particular solution.

This is also an example where we should break up $g(t)$ as $g(t) = g_1(t) + g_2(t)$, and solve each sub-problem separately.

First, solve the characteristic equation: $r^2 + 2r = 0$, so $r = 0$ and $r = -2$. Therefore, the homogeneous part of the solution is

$$y_h(t) = C_1 + C_2 e^{-2t}$$

Now, to solve for the particular solution, take $g_1(t) = 3$ first. Normally, we would guess that the particular solution is $y_{p_1} = A$, but since the homogenenous solution has a constant solution, multiply the particular solution by t, then solve:

$$\begin{aligned} y_{p_1} &= At \\ y_{p_1}' &= A \\ y_{p_1}'' &= \end{aligned} \qquad \Rightarrow \qquad 0 + 2A = 3 \qquad \Rightarrow \qquad A = \frac{3}{2}$$

Therefore, $y_{p_1}(t) = \frac{3}{2}t$.

For the second, $y_{p_2} = A\cos(2t) + B\sin(2t)$, and substitution yields the following (for bookkeeping, I try to line things up in the following way):

$$\begin{array}{rlll} 0(y & = A\cos(2t) & +B\sin(2t)) & \\ 2(y' & = 2B\cos(2t) & -2A\sin(2t)) & \\ y'' & = -4A\cos(2t) & -4B\sin(2t) & \\ \hline y'' + 2y' & = (4B - 4A)\cos(2t) & -(4A + 4B)\sin(2t) & = 4\sin(2t) \end{array}$$

From this, we get two equations: $4A - 4B = 0$ and $-4A - 4B = 4$. Solving, we get $A = B = -1/2$.

Therefore, the overall solution is:

$$y(t) = C_1 + C_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)$$

6. In this one, we get a repeated root $r = -1, -1$. Therefore, the homogeneous part of the solution is:

$$y_h(t) = C_1 e^{-t} + C_2 t e^{-t}$$

Looking at $g(t) = 2e^{-t}$, we might initially guess $y_p = Ae^{-t}$, but that is already part of y_h. Therefore, multiply by t to get a guess of $y_p = Ate^{-t}$, but that is also already part of y_h. Finally, multiply by t again to get the final form of our particular solution,

$$y_p = At^2 e^{-t} \qquad y_p' = (-t^2 + 2t)e^{-t} \qquad y_p'' = (t^2 - 4t + 2)e^{-t}$$

And substitution yields $A = 1$. Therefore, the overall solution is:

$$y = e^{-t}\left(C_1 + C_2 t + t^2\right)$$

7. This is another exercise where we want to "pull apart" the forcing function $g(t)$ into two pieces: $g(t) = g_1(t) + g_2(t) = t^2 + 3\sin(t)$.

Side Remark: This is because t^2 and $\sin(t)$ are distinctive types of functions- If this had been $\sin(t)$ and $\cos(t)$, we would NOT have split them up.

Solving the characteristic equation, we get $r = -1$ and $-1/2$, so the homogeneous part of the solution is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-t/2}$$

For $g_1(t) = t^2$, our ansatz is $y_{p_1}(t) = At^2 + Bt + C$ (and no part of this solution is contained within the homogeneous solution). Substitute this into the DE, and solve for the coefficients. When you do this, you should see that you are solving the following three equations:

$$A = 1 \qquad B + 6A = 0 \qquad 4A + 3B + C = 0$$

From which we get $y_{p_1} = t^2 - 6t + 14$. For the next one, we take $y_{p_2}(t) = A\cos(t) + B\sin(t)$, and perform the usual computations.

9, 10. We want to compare and contrast these solutions-

- They both have the same characteristic equation and homogeneous part of the solution:

$$y_h = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$

- In Exercise 9, the particular solution would have the form:

$$y_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

But in Exercise 10, we have to multiply by t

$$y_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

This difference will have some implications later.

- Finally, solve for the coefficients- It's a lot easier to do in 9 than in 10, but give it a shot (Answers in the back of the text).

For Exercises 19-26, only state the form of the solution. Optional: Check your answers using Wolfram Alpha- The commands you would use are given below.

19. Roots are $r = 0, -3$ and there is some overlap with the homogeneous solution. In this case, we will also parse out the particular solution (into 3 pieces):

$$y_h = C_1 + C_2\mathrm{e}^{-3t}$$

For $g_1(t) = 2t^4$, we have a generic 4th degree polynomial- But constant solutions are in y_h, so multiply by t:

$$y_{p_1} = t(At^4 + Bt^3 + Ct^2 + Dt + E)$$

For $g_2(t) = t^2\mathrm{e}^{-3t}$, our initial guess would be a (full) quadratic times the exponential, but a constant times the exponential is part of y_h, so multiply by t:

$$y_{p_2} = t(Ft^2 + Gt + H)\mathrm{e}^{-3t}$$

And finally, $g_3(t) = \sin(3t)$, so

$$y_{p_3} = I\cos(3t) + J\sin(3t)$$

(NOTE: Try to not double up on your arbitrary constants, especially if you're putting everything together at the end).

The overall solution from Wolfram Alpha is:

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solve y''+3y'=2t^4+t^2exp(-3t)+sin(3t)
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And in fact, if you press the "Show Steps" button, will give you the steps. Therefore, I won't include solutions for the remaining questions in this section.

28. This is a nice problem in that we exploit the structure of the particular solution- That is, we solve $L(y) = g_1(t)$, then solve $L(y) = g_2(t)$, etc, then add the solutions up. In this case, we can write

$$y'' + \lambda^2 y = a_1 \sin(\pi t) + a_2 \sin(2\pi t) + a_3 \sin(3\pi t) + \cdots + a_N \sin(N\pi t)$$

If $\lambda \neq \pi, 2\pi, 3\pi, \cdots, N\pi$, then the homogeneous part of the solution and the particular parts of the solution are distinct (no overlap), so we don't need to multiply by t. In fact, if

$$g_m(t) = a_m \sin(m\pi t) \quad \Rightarrow \quad y_{p_m}(t) = A_m \cos(m\pi t) + B_m \sin(m\pi t)$$

Put this back into the DE to get:

$$\begin{array}{rcrl} \lambda^2 y_{p_m} & = & \lambda^2 A_m \cos(m\pi t) & +\lambda^2 B_m \sin(m\pi t) \\ y''_{p_m} & = & -A_m m^2\pi^2 \cos(m\pi t) & -B_m m^2\pi^2 \sin(m\pi t) \\ \hline a_m \sin(m\pi t) & = & A_m(\lambda^2 - m^2\pi^2)\cos(m\pi t) & +B_m(\lambda^2 - m^2\pi^2)\sin(m\pi t) \end{array}$$

From which we see that $A_m = 0$ and

$$B_m = \frac{a_m}{\lambda^2 - m^2\pi^2}$$

Therefore, the solution is:

$$y(t) = C_1 \cos(\lambda t) + C_2 \sin(\lambda t) + \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin(m\pi t)$$

In 34 and 35, we see a very interesting approach to solving the non-homogeneous DE using the operator notation. Exercise 33 gives the details, but leaves off one detail: How should we solve

$$(D + a)x(t) = g(t)$$

for x? This is linear with an integrating factor. We see that the solution is:

$$x(t) = \mathrm{e}^{-at} \int g(t)\mathrm{e}^{at}\, dt$$

Now we can use this to speed things up below:

34. (35 is similar)

$$y'' - 3y' - 4y = (D^2 - 3D - 4)y = (D + 1)(D - 4)y = 3\mathrm{e}^{2t}$$

First we solve $(D + 1)u(t) = 3\mathrm{e}^{2t}$:

$$u(t) = \mathrm{e}^{-t} \int 3\mathrm{e}^{2t}\mathrm{e}^{t} = \mathrm{e}^{-t}\left(\mathrm{e}^{3t} + C_1\right) = \mathrm{e}^{2t} + C_1\mathrm{e}^{-t}$$

Now solve $(D-4)y(t) = \mathrm{e}^{2t} + C_1\mathrm{e}^{-t}$:

$$y(t) = \mathrm{e}^{4t} \int \left(\mathrm{e}^{2t} + C_1\mathrm{e}^{-t}\right) \mathrm{e}^{-4t}\, dt = \mathrm{e}^{-4t} \left(-\frac{1}{2}\mathrm{e}^{-2t} - \frac{C_1}{5}\mathrm{e}^{-5t} + C_2\right) \Rightarrow$$

$$y(t) = -\frac{1}{2}\mathrm{e}^{2t} + \hat{C}_1\mathrm{e}^{-t} + C_2\mathrm{e}^{4t}$$